

ABSTRACT

LEPTON FLAVOR VIOLATING RADION DECAYS IN THE RANDALL-SUNDRUM SCENARIO: THE THESIS

Korutlu, Beste

M. Sc., Department of Physics

Supervisor: Dr. Erhan Onur İltan

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The lepton flavor violating interactions are worthwhile to examine since they are sensitive to physics beyond the Standard Model. The simplest extension of the Standard Model promoting the lepton flavor violating interactions are the so called two Higgs doublet model which contains an additional Higgs doublet carrying the same quantum numbers as the first one. In this model, the lepton flavor violating interactions are induced by new scalar Higgs bosons, scalar h^0 and pseudo scalar A^0 , and Yukawa couplings, appearing as free parameters, are determined by using the experimental data. On the other hand, the possible extra dimensions are interesting in the sense that they ensure a solution to the hierarchy and cosmological constant problems and also result in the enhancement in the physical quantities of various processes. In the present work, we predict the branching ratios of lepton flavor violating radion decays $r \rightarrow e^\pm, \mu^\pm$, $r \rightarrow e^\pm, \tau^\pm$ and $r \rightarrow \mu^\pm, \tau^\pm$ in the two Higgs doublet model, including a single extra dimension, in the framework of the Randall Sundrum scenario. We observed that the branching ratios of the processes we study are at most at the order of 10^{-8} for the small values of radion mass and it decreases with the increasing values of the radion mass. Among the LFV decays we study, the $r \rightarrow \mu^\pm, \tau^\pm$ decay would be the most suitable one to measure its branching ratio.

Keywords: Standard Model, Lepton Flavor Violation, Branching Ratio, Radion,

Two Higgs Doublet Model, Randall Sundrum Scenario.

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to my lovely family..

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CHAPTER 1

INTRODUCTION

The major goal of physics has always been the simplification and the unification of seemingly diverse and complicated natural phenomena. In the second half of the twentieth century, as a result of successful approaches, a significant progress has been made in the particle physics, in the identification of fundamental particles and the unification of their interactions. Glashow-Weinberg-Salam [1, 2] combined the quantum electrodynamics (QED) and the weak interactions into the electroweak (EW) theory and the Standard Model (SM) of elementary particles has emerged, which can be considered as a good example satisfying this major goal of physics. Being a quantum field theory (QFT) based on the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, the SM describes all of the known elementary constituents of the universe together with the three out of four fundamental forces: the strong force, the weak force, and the electromagnetic force. In the QFT, all interactions are mediated by means of force carrier particles, mediators. In the case of electromagnetic interaction the mediator is the photon (γ), one of the four gauge bosons of the group $SU(2)_L \otimes U(1)_Y$, and for the weak interactions, there exist the remaining gauge bosons of the same group, the so called W^\pm , Z^0 bosons. In addition to this, for the strong interactions, the mediators are eight gluons (G_i), the gauge bosons of the group $SU(3)_C$. The remaining force, called as the gravity, is far too weak to be of any consequence at the experimentally accessible energy scales that are relevant to particle physics. In addition to the mediator particles, the SM contains matter particles: the Higgs boson and the fermions, namely leptons and quarks which fall into three generations. The first generation contains all stable stuff of which the stable matter is composed. The second and third generations of particles decay, therefore, they are not present in the stable matter and physicists are still trying to understand their role in the

underlying theory. For each quark and lepton there exists a corresponding anti-quark and antilepton. This is all adding up to an embarrassingly large number of elementary particles: 12 leptons, 36 quarks, 12 mediators, and, as we will see later, Glashow-Weinberg-Salam theory calls for at least one Higgs particle, so we have a minimum of 61 particles to content with. In the next chapter, we will see how this structure leads to the first consistent and self-contained theory. The energy range which defines this theory extends up to several hundreds GeV. For the details of the model construction see for example textbooks [3, 4], and the review [5] existing in the literature.

The SM has been very successful in explaining many diverse experimental results in the energy range available at present. However, behind this energy range it possesses some conceptual problems which motivate us to look physics beyond. The big issues in physics beyond the SM can be conveniently grouped into four categories. The Unification: What is the reason beyond the hierarchy of fundamental forces? The problem of Flavor: Why are there so many different types of quarks and leptons? The Mass problem: What is the origin of masses of fundamental particles and their mass hierarchies? Does the Higgs boson exist? The cosmological constant problem. In addition to the conceptual problems of SM, there also exist phenomenological hints obtained from measurements of flavor changing neutral currents (FCNCs), including lepton flavor violating (LFV) interactions which also indicate the need for physics beyond the SM since the SM predictions differ from the upper limits coming from current measurements.

There are various alternative extended models proposed for solving these problems of the SM such as the multi Higgs doublet model (MHDM) [6, 7, 8, 9], the minimal supersymmetric model (MSSM) [10, 11, 12, 13], left-right (super) symmetric model [14], the Zee Model [15], the see-saw model [16], technicolor model [17], extra dimensional models [18]-[31]; large extra dimensions [19, 20, 21], universal extra dimensions (UED) [22, 23, 24], non-universal extra dimensions (NUED) [25, 26], split fermion scenarios [27, 28, 29], the Randall-Sundrum model (RS model) [30, 31].

Based on the phenomenological hints, the violation of flavor symmetry in the leptonic sector is of special interest to physicists. In the SM, the FCNCs

with massless neutrino, are not allowed in the lepton sector and, in the quark sector, they are prohibited at tree level, despite they seem not to violate any fundamental law of nature. The negligibly small branching ratios (BRs) of the decays based on the FCNCs stimulate one to go beyond the SM and they are worthwhile to examine since they open a window to test new models, to ensure considerable information about the restrictions of the free parameters, with the help of the possible accurate measurements. An elegant framework to open up the possibility of the tree level FCNCs is proposed through the general two Higgs doublet model (2HDM) (see [7, 8, 9] for details), the most primitive candidate of MHDM, which is obtained by adding a second Higgs doublet, having the same quantum numbers as the first one. This doublet may lead to FCNCs in its Yukawa sector, representing interactions between the Higgs fields and fermions (see for example [9]). In this model, the lepton flavor (LF) violation is driven by the new scalar. In addition, the mass hierarchy problem among third generation of quarks, namely the top and bottom quarks, also could be solved in the scope of 2HDM, such that, unlike in the SM where both quarks gain mass through the interaction with the same Higgs doublet, there is a possibility that the bottom receives its mass from one doublet (say ϕ_1) and the top from the another one (say ϕ_2). Then the hierarchy of their Yukawa couplings could be more natural.

A theory which consists of the SM, combined with gravity, contains two enormously different energy scales. One is the EW scale $m_{EW} \sim 10^3$ GeV at which EW symmetry is broken, and the other is the Planck scale $M_{Pl} \sim 10^{19}$ GeV which determines the strength of gravitational interactions. Newton's laws state that the strength is inversely proportional to the second power of that energy, and because the strength of gravity is so small, the Planck scale mass (related to the Planck scale energy by $E = mc^2$) should be very large. Generally, when making predictions in particle physics, we can ignore gravity since the gravitational effects on particles in the EW energy scale are completely negligible. But that is precisely a question which particle physicists try to find an answer: Why is the gravity so weak? A solution to this problem comes from models with extra dimensions where gravity becomes strong and cannot be neglected. In 1998, Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali [20, 21] proposed a model

(called as ADD Model) with n compact extra spatial dimensions of large size to bring the Planck scale down to TeV scale. Depending on the details of their implementation, the space in their model contains two, three or more compact extra dimensions. For two extra dimensions, the hierarchy problem in the fundamental scales could be solved and the true scale of quantum gravity would be no more the Planck scale but of the order of EW scale. This is the case that the gravity is spreading over all the volume including the extra dimensions and thus it is diluted by a large volume of them so much that it will be very feeble in the lower dimensional effective theory¹, although it is very strong in higher dimensions. On the other hand, the matter fields together with the electromagnetic, strong and weak forces are restricted in four dimensions, called four dimensional (4D) brane. Unlike gravitational force, these forces will not be accessible to the higher dimensions. As mentioned above, in ADD model, the extra dimensions are compact and their compactification leads to the appearance of towers of heavy Kaluza-Klein (KK) modes [32] of particles such that, in 4D effective theory, the existence of the extra dimensions are felt by the appearance of these KK modes. However, since the matter fields do not travel along the extra dimensions but bound to 4D brane, they will not carry extra dimensional momenta. In other words, none of the SM particles will have the KK partners. The only particle that will have KK partners is the graviton, the force carrier particle of the gravitational force. Since the KK partners of graviton also interact with gravitational strength (i.e., as weakly as graviton itself), it would be no easier to produce or detect KK partners of the graviton than to observe the graviton itself which also has never directly seen by anyone up to now. This means that, gravity, being the only force which lives in higher dimensions, the existence of large extra dimensions will not contradict with the experimental results. Despite the success of ADD proposal in solving the hierarchy problem, there exist also some weaknesses of the theory. In fact their model do not actually solve the hierarchy problem, because one still have to solve why the size of extra dimensions are so large.

An alternative approach is introduced by Randall and Sundrum (RS1 model)

¹Effective theory is a theory describing those elements and forces that are in principle observable at the distance or energy scales over which it is applied.

[30, 31] to explain the huge discrepancy between m_{EW} and M_{Pl} without the need for a large extra dimension, or for any arbitrary large number at all. In this scenario, the geometry is a non-factorizable one where the gravity is localized in a 4D brane, so called Planck brane, which is one of the boundary of the extra dimension and away from another 4D brane, TeV brane, which is the other boundary where we live². Theory also includes a finely tuned 5D cosmological constant Λ which serve as sources for 5D gravity and in 4D, with the help of opposite tensions on boundaries, it vanishes.

The review topics we include in this thesis is, broadly, divided into three categories: In Chapter 2, we give a brief review of the SM. Chapter 3 is devoted to the simplest extension of the SM, the so called the 2HDM. In Chapter 4, we give a summary of models with extra dimensions. In Chapter 5, we investigate the branching ratios of lepton flavor violating radion decays $r \rightarrow e^\pm, \mu^\pm$, $r \rightarrow e^\pm, \tau^\pm$ and $r \rightarrow \mu^\pm, \tau^\pm$ in the 2HDM, in the framework of the Randall Sundrum scenario (RS1). Chapter 6 represents our conclusions. In Appendix A, we present the global and local gauge invariance. Appendix B is devoted to the detailed calculations of Einstein equations, that we use in our review. Appendix C represents the calculation of spin connection.

²The extra dimension is compactified to S^1/Z_2 orbifold with two 4D brane boundaries which reside at the orbifold fixed points.

CHAPTER 2

THE STANDARD MODEL

In the second half of the twentieth century physicists made an impressive contribution to the progresses in particle physics with complementary theoretical and experimental studies, whereupon the SM of elementary particles has emerged [1, 2]. The SM, being a QFT (see for example [33]), describes all of the known elementary particles together with the three out of four fundamental interactions of nature. According to the SM, the elementary particles constituting the universe are called as fermions (i.e., they have spin one-half), namely quarks and leptons and the fundamental interactions are the electromagnetic force, the weak force (responsible for radioactive decay) and the strong force (which holds atomic nuclei together). The SM is based on the principle of gauge symmetry (see Appendix A), which means that the properties and interactions of elementary particles are governed by certain symmetries which are related to the conservation laws. Therefore, the electromagnetic, weak, and strong forces are all gauge forces and they are mediated by the exchange of certain particles, called gauge bosons (i.e., they have spin one) which are the photon, the W^\pm and Z^0 bosons, and eight gluons, respectively. Several attempts have been made to fit gravity, the remaining force, into this gauge framework but these attempts are resulted in failure. However, the gravity is too weak to change particle physics predictions in the current experimental energy scales.

With these in mind, it is worthwhile summarizing the Fermi theory [34] which describes the weak interaction phenomenology in the mid-1950.

2.1 The Fermi Theory

The progress in the field theory of the weak interaction was rather stagnant for many decades, from Fermi's attempt to describe the β decay

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad (2.1)$$

in 1933, to the advent of the gauge theories in the 1970s. Fermi expressed this decay mathematically as, at a single point in the space-time, the quantum mechanical wavefunction of a neutron is transformed into the wavefunction of a proton, and that the wavefunction of an incoming neutrino is transformed into that of an electron. He wrote the phenomenological Lagrangian as

$$\mathcal{L}_F = \frac{G_F}{\sqrt{2}} J_{\alpha, had}^\dagger(x) J_{lept}^\alpha(x) + h.c., \quad (2.2)$$

where $G_F = (1.16639 \pm 0.00002) \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi coupling constant, $J_{\alpha, had}^\dagger(x) = \bar{\psi}_p(x) \Gamma_\alpha \psi_n(x)$ and $J_{lept}^\alpha(x) = \bar{\psi}_e(x) \Gamma^\alpha \psi_{\nu_e}(x)$. This action, from very start, was known to suffer from a series of problems. First of all, the Γ_α matrices, that contain the essence of the weak interaction, consist of all possible combinations of the 16 Dirac matrices. It took many years to narrow down the choice. In 1958, Feynman and Gell-Mann [35] with the help of further experimental data proposed that the correct combination of Γ_α matrices should only contain a mixture of vector and axial-vector¹ (V-A) quantities written in the form $\Gamma_\alpha = \gamma_\alpha(1 - \gamma_5)$ to incorporate the parity-violating effects of the weak interaction.

Since the weak force is of extremely short range, Fermi's theory of point-like interaction yields excellent approximate results at low energies. However, at high energies the theory suffers from additional problems. The main problem is the violation of unitarity. $\nu_e + e^- \rightarrow \nu_e + e^-$ scattering is one of the simplest example of the weak interaction processes. The Feynman diagram for this scattering in the Fermi's picture of four point interaction is given in the Fig. 2.1. In the

¹an axial vector (or a pseudovector) is a quantity that transforms like a vector under a proper rotation, but gains an additional sign flip under an improper rotation (a transformation that can be expressed as an inversion followed by a proper rotation).

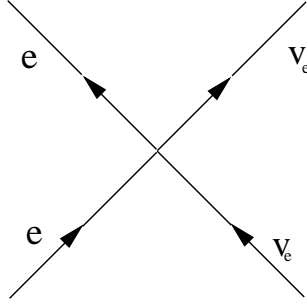


Figure 2.1: The four point $\nu_e + e^- \rightarrow \nu_e + e^-$ scattering.

center of mass frame (CM), the differential cross section is found as

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 k^2}{\pi^2}, \quad (2.3)$$

with CM four momentum k , and

$$\sigma = \frac{4G_F^2 k^2}{\pi}, \quad (2.4)$$

where $k^2 \gg m_e^2$. Since the four fermion interaction takes place at a single point in space-time, the differential cross section is a pure s-wave. Partial wave unitarity for s-wave requires that

$$\sigma < \frac{\pi}{2k^2}. \quad (2.5)$$

Using the above equation, k is obtained as

$$k^4 < \frac{\pi^2}{8G_F^2}. \quad (2.6)$$

Then, above a certain energy (i.e., $k > 300$ GeV), Fermi theory violates unitarity. Thus, we can say that it is an effective theory up to the energies $k < 300$ GeV.

Another problem in this theory is the non-renormalizability. Unfortunately, even in the lowest order approximation in four fermion interactions, one encounters horrible divergences which cannot be eliminated by proper renormalization. The Feynman diagram of the four point interaction including the lowest order correction for the scattering $\nu_e + e^- \rightarrow \nu_e + e^-$ is shown in the Fig. 2.2 below

which is generated by multiplying the current-current interaction with itself and the amplitude for this scattering is proportional to $\propto \frac{d^4 k}{k^2} = \infty^2$. To eliminate

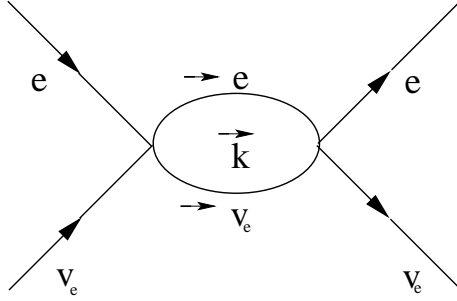


Figure 2.2: One loop correction to the $\nu_e + e^- \rightarrow \nu_e + e^-$ scattering .

these problems, the idea is that the weak interaction is mediated by intermediate massive vector boson exchange. The Feynman diagram for this process is shown in the Fig. 2.3. Therefore, we replace \mathcal{L}_F defined in the eq. 2.2 with

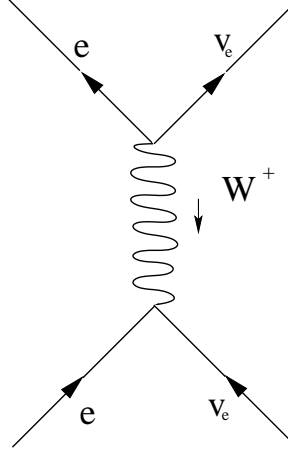


Figure 2.3: The $\nu_e + e^- \rightarrow \nu_e + e^-$ scattering mediated by intermediate massive vector boson exchange.

$$\mathcal{L}_W = g_W J^\alpha(x) W_\alpha(x) + h.c., \quad (2.7)$$

where $W_\alpha(x)$ is the weak intermediate vector boson. Now, in the lowest order diagram the differential cross section of the $\nu_e + e^- \rightarrow \nu_e + e^-$ scattering is found

as

$$\frac{d\sigma}{d\Omega} = \frac{2g_W^4 k^2}{\pi^2(q^2 - m_W^2)^2}, \quad (2.8)$$

for again $k^2 \geq m_e^2$. Here, m_W and q are the mass and the momentum transfer vector of the W boson, respectively. As $q^2 \rightarrow 0$, the new Lagrangian in the eq. 2.7 reduces to Fermi Lagrangian given in the eq. 2.2 provided

$$\frac{g_W^2}{m_W^2} = \frac{G_F}{\sqrt{2}}. \quad (2.9)$$

However, the interaction is no longer point-like, but mediated by the force carrier particles. In the scope of the SM, all fundamental interactions are mediated by the exchange of gauge bosons. These interactions together with the corresponding gauge bosons which mediate the forces are listed in the following table in order of decreasing strength.

Table 2.1: The four fundamental forces in nature.

Force	Strength	Range	Theory	Mediator
Strong	10	$< 10^{-15}m$	Chromodynamics	Gluon
Electromagnetic	10^{-2}	∞	Electrodynamics	Photon
Weak	10^{-13}	$< 10^{-18}m$	Flavordynamics	W^\pm, Z
Gravitational	10^{-42}	∞	Geometrodynamics	Graviton

According to the QFT, the short range of the weak force could mean only one thing: the weak gauge bosons had to have non zero masses. The mechanism that gives rise to the masses of gauge bosons is known as the Higgs mechanism [36] which relies on the phenomenon of spontaneous symmetry breaking (SSB) which we will consider in the following section.

2.2 Spontaneously Broken Symmetries

Symmetry is one of the most important aspects of theoretical particle physics, since the basis of our current description of nature originate in symmetries so

that every continuous symmetry leads to a conservation law. A system is said to be symmetric if it remains invariant after applying a set of transformation rules that constitute a mathematical group. Symmetries are categorized into two: spatial symmetry and internal symmetry. In the case of spatial symmetry physics treats all directions and all positions as the same, internal symmetries tell us that physical laws act the same way on distinct, but effectively indistinguishable objects. The fundamental forces, electromagnetic, weak, and strong forces all involve internal symmetries. (Gravity is related to the symmetries of space and time). Exact symmetries are fairly rare in nature. Thus, the symmetries that the usual 4D theories possess can be broken explicitly or spontaneously. In the case of SSB of gauge symmetries, which is one of the crucial ingredients of the SM, if the broken symmetry is global, the Goldstone theorem [37] applies, whereas if it is local, then we have Higgs mechanism [36]. In general, the phenomenon of SSB is simply stated as follows.

“A system is said to possess a symmetry that is spontaneously broken if the ground state of a dynamical system does not possess the same symmetry properties as the Lagrangian”. Here, the ground state -its vacuum state- is the state in which the field has its lowest possible energy. Now, we will use a toy model (see [4] for details) to explain the Goldstone theorem and the Higgs mechanism.

2.2.1 The Goldstone theorem

Let us consider the following Lagrangian density which describes a couple of self interacting complex scalar fields $\phi(x) = \phi_1(x) + i\phi_2(x)$ and its complex conjugate $\phi^*(x) = \phi_1^*(x) - i\phi_2^*(x)$

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - \mu^2 \phi \phi^* - \lambda(\phi \phi^*)^2, \quad (2.10)$$

where μ^2 is regarded as the bare mass of the field quanta and λ is the term for self interaction. It is clear that the Lagrangian remains invariant under the group $U(1)$ of global gauge transformations (see Appendix A for details). For SSB, we should check whether the ground state of the system will be invariant

under global gauge transformations or not. For constant ϕ the kinetic term, $(\partial_\mu \phi)(\partial^\mu \phi^*)$ vanishes. Then, the ground state is obtained when the potential term $V(\phi, \phi^*) = \mu^2 \phi \phi^* + \lambda(\phi \phi^*)^2$ corresponds to the minimum. Since the potential term is a function of ϕ and ϕ^* only in the combination of $\phi \phi^*$, we can make a change of variables so that, $\rho = \phi \phi^*$. Substituting this into $V(\phi, \phi^*)$ we get

$$V(\rho) = \mu^2 \rho + \lambda \rho^2. \quad (2.11)$$

The minimum of the potential can be obtained only if $\lambda > 0$, which we take to be so. However, μ^2 can have both positive and negative values if we do not insist on interpreting μ as mass. To find the minimum of the potential, we take the derivative of $V(\rho)$ with respect to ρ and equate this derivative to zero such that,

$$\frac{dV(\rho)}{d\rho} = \mu^2 + 2\lambda\rho = 0. \quad (2.12)$$

Since $\rho = \phi \phi^* = (\phi_1 + i\phi_2)(\phi_1^* - i\phi_2^*) = (|\phi_1|^2 + |\phi_2|^2)$, ρ can only take positive values. Therefore, for $\mu^2 > 0$ a unique minimum occurs at the origin $\rho = 0$, i.e., at $\phi = 0$ and the function $V(\rho)$ looks like as in figure 2.4. Then, we have

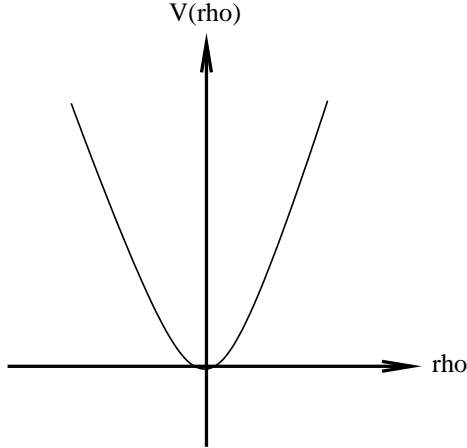


Figure 2.4: The potential function for positive μ^2 .

a symmetric ground state configuration under the group $U(1)$ of global gauge transformations for $\mu^2 > 0$. On the other hand, for $\mu^2 < 0$, $\phi = 0$ is not a

minimum. Instead, the minimum is at $\rho = -\mu^2/2\lambda$, i.e., at $|\phi| = v/\sqrt{2}$ with $v = \sqrt{-\mu^2/\lambda}$. Any value of ϕ satisfying this relation will give us a true ground state such that

$$\phi_{vac} = \frac{v}{\sqrt{2}}e^{i\Lambda}, \quad (2.13)$$

where Λ is real. Then, we have a continuum degenerate set of ground states for negative values of μ^2 . In this case, the function $V(\phi)$ looks like as in figure 2.5. Each will not be symmetric under the global gauge transformation defined in the

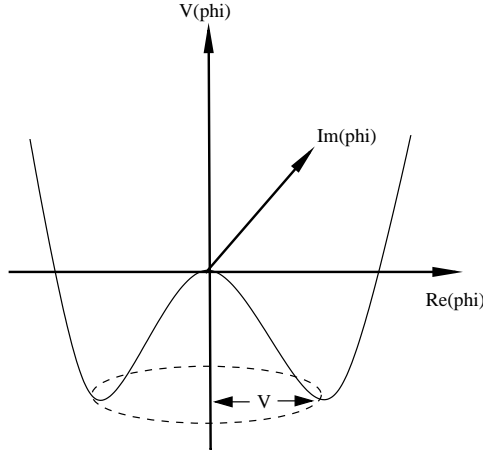


Figure 2.5: The potential function for negative μ^2 .

eq. A-2. Then, using the definition of SSB made above one can simply conclude that the symmetry of the Lagrangian has been spontaneously broken. We are free to choose any point on the ring of minima since they are equivalent. Let us choose this point to be on the real axis such that

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \xi(x) + i\chi(x)], \quad (2.14)$$

where $\xi(x)$, $\chi(x)$ are real fields and $\xi(x) = \chi(x) = 0$ in the ground state. Substituting into the eq. 2.10 and ignoring constant terms we get

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial^\mu \chi)^2 - \lambda v^2 \xi^2 - \lambda v \xi(\xi^2 + \chi^2) - \frac{1}{4}\lambda(\xi^2 + \chi^2)^2. \quad (2.15)$$

Then, we end up with a massless $\chi(x)$ field and a field $\xi(x)$ with a spontaneously generated mass

$$m_\xi(x) = 2\lambda v^2. \quad (2.16)$$

Let us now examine a less trivial example [5] given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{1}{2}\mu^2\phi^i\phi^i - \frac{1}{4}\lambda(\phi^i\phi^i)^2, \quad (2.17)$$

where ϕ is an n -component real scalar field and \mathcal{L} is invariant under the orthogonal group in n dimensions, $O(n)$. Again for $\mu^2 < 0$ we find a whole ring of minima whenever $\sum_i \phi^i\phi^i = -\mu^2/\lambda$ is satisfied. In this case, we are free to choose one of the ϕ^i to be non-zero in the ground state. Let it be the n^{th} component of ϕ such that,

$$\phi_{vac} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ v \end{pmatrix}. \quad (2.18)$$

The number of generators that original symmetry group $O(n)$ possesses is $\frac{1}{2}n(n-1)$. There is also a non-trivial subgroup $O(n-1)$, which has $\frac{1}{2}(n-1)(n-2)$ number of generators leave the vacuum invariant. Let L_{ij} be the $\frac{1}{2}n(n-1)$ independent matrices that generates $O(n)$ and $l_{ij}[l_{ij} = L_{ij} \text{ for } i, j \neq n]$ be the $\frac{1}{2}(n-1)(n-2)$ matrices generating $O(n-1)$. There remains $n-1$ independent matrices which are denoted by $k_i[k_i = L_{in}]$ with $1 \leq i \leq n-1$. Defining η and ξ_i with again

$1 \leq i \leq n-1$ we get

$$\phi = e^{(i\xi_i k_i/v)} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v + \eta \end{pmatrix}. \quad (2.19)$$

Note that in general,

$$(L_{ij})_{kl} = -i[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]. \quad (2.20)$$

For $j=n$ we have

$$(L_{in})_{kl} = (k_i)_{kl} = -i[\delta_{ik}\delta_{nl} - \delta_{il}\delta_{nk}]. \quad (2.21)$$

Operating k_i on the column vector $v_i = v\delta_{in}$ we get

$$\begin{aligned} (k_i v)_j &= (k_i)_{jl} v_l \\ &= -i[\delta_{ij}\delta_{nl} - \delta_{il}\delta_{nj}] v_l \\ &= -i[\delta_{ij}v_n - \delta_{nj}v_i] \\ &= -i[\delta_{ij}v\delta_{nn} - \delta_{nj}v\delta_{in}] \\ &= -iv\delta_{ij}. \end{aligned} \quad (2.22)$$

Thus, in the lowest order $\phi_i = \xi_i (i < n)$ and $\phi_n = v + \eta$. Then, the Lagrangian density in terms of ξ_i and η can be presented as

$$\mathcal{L} = \frac{1}{2}[\partial^\mu \eta \partial_\mu \eta] + \partial^\mu \xi_i \partial_\mu \xi_i - \frac{1}{2}\mu^2(v + \eta)^2 - \lambda(v + \eta)^4 + \dots \quad (2.23)$$

Looking at this Lagrangian, we can say that the field η has a positive mass of $-2\mu^2$ and the $(n-1)$ ξ_i scalar fields remain massless. These massless bosons are called as Goldstone bosons. In conclusion, for every broken generator that leaves the vacuum invariant there exists a massless Goldstone boson.

2.3 The Higgs Mechanism

Now, we will use the same Lagrangian in the eq. 2.10 but impose invariance under $U(1)$ of local gauge transformations (see Appendix A for details). To make the Lagrangian invariant under this transformation we must replace the partial derivative ∂_μ by the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ and add a kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Then, the Lagrangian density becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + [(\partial^\mu + ieA^\mu)\phi^*(\partial_\mu - ieA_\mu)\phi] - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2, \quad (2.24)$$

where A_μ is the massless gauge field. Under local gauge transformations we have,

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = \exp^{-i\theta(x)} \phi(x), \\ \phi^*(x) &\rightarrow \phi^{*'}(x) = \exp^{i\theta(x)} \phi^*(x), \\ A_\mu &\rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x). \end{aligned} \quad (2.25)$$

Again we will look at the minimum in the potential. For $\lambda \geq 0$ and $\mu^2 > 0$ we obtain a symmetric ground state at $\phi = 0$. However, when $\mu^2 < 0$ there exists again a ring of degenerate ground states. Proceeding as before we set

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \xi(x) + i\chi(x)], \quad (2.26)$$

with $v = \sqrt{-\mu^2/\lambda}$ so that $\phi_{vac} = v/\sqrt{2}$. Substituting this into 2.24, we obtain the following Lagrangian density

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2v^2}{2}A_\mu A^\mu + \frac{1}{2}(\partial^\mu\xi)^2 + \frac{1}{2}(\partial^\mu\chi)^2 \\ & -\frac{1}{2}(2\lambda v^2)\xi^2 - evA_\mu\partial^\mu\chi + \dots \end{aligned} \quad (2.27)$$

It is surprising that the gauge field A_μ seems to acquire mass in the quantum picture. The Lagrangian density in the eq. 2.27 now seems to describe the interaction of a massive gauge field A_μ and two scalar fields. To ensure the gauge invariance, the gauge transformations in terms of $\xi(x)$ and $\chi(x)$ should be in the

following form

$$\begin{aligned}
\xi(x) &\rightarrow \xi'(x) = [v + \xi(x)] \cos \theta(x) + \chi(x) \sin \theta(x) - v, \\
\chi(x) &\rightarrow \chi'(x) = \chi(x) \cos \theta(x) - [v + \xi(x)] \sin \theta(x), \\
A_\mu &\rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \theta(x).
\end{aligned} \tag{2.28}$$

We are free to choose $\theta(x)$ to be the phase of $\phi(x)$ since the theory is invariant under any choice of transformation of this function. Then,

$$\phi'(x) = \exp^{-i\theta(x)} \phi(x) = \frac{1}{\sqrt{2}} [v + \eta(x)], \tag{2.29}$$

will be real, with $\eta(x)$ is real. Substituting these into the eq. 2.24 the Lagrangian density becomes

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} + \frac{1}{2} \partial^\mu \eta \partial_\mu \eta + \frac{1}{2} e^2 v^2 A'_\mu A'^\mu \\
& + \frac{1}{2} e^2 (A'_\mu)^2 (2v\eta + \eta^2) - \lambda v^2 \eta^2 - \frac{1}{4} \lambda \eta^4 \dots
\end{aligned} \tag{2.30}$$

with $F'_{\mu\nu} F'^{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$. By writing the Lagrangian density in this form, we can say that it describes the interaction of the massive vector field A'_μ with the massive, real, scalar field η . This field is called as Higgs field with a mass of $2\lambda v^2 = -2\mu^2$. In this way, all massless particles completely disappears. Consequently, in spontaneously broken symmetries the gauge boson acquires mass due to disappearing Goldstone boson. Therefore, for each massive gauge fields we need a complex scalar field, one piece of which disappears and reappears as the longitudinal mode of the vector field. Scalar part of this complex field, the so called Higgs boson, remains.

2.4 The Standard Model Lagrangian

Having discussed the ingredients of the SM, let us turn our attention to the SM Lagrangian, \mathcal{L}_{SM} . As mentioned above the SM is based on the principle of gauge symmetry. The overall gauge group of the SM, under which the SM Lagrangian

remains invariant, contains both the Quantum Chromodynamics (QCD) and the unified EW interaction and is written symbolically as

$$G_{SM} \equiv SU(3)_C \otimes SU(2)_L \otimes U(1)_Y. \quad (2.31)$$

The first group, $SU(3)_C$, represents QCD. The subscript C indicates that the gauge bosons of QCD, the eight gluons, couple only to colored particles, quarks. The remaining part $SU(2)_L \times U(1)_Y$ represents the EW interaction, proposed by Glashow-Weinberg-Salam [1, 2], with the subscripts L and Y indicating that the group $SU(2)_L$ couples only left handed particles and that the group $U(1)_Y$ couples to weak hypercharged particles where the hypercharge is obtained using the Gell-Mann-Nishijima relation [38] $Q = T_3 + Y/2$. The EW theory, developed by Glashow-Weinberg-Salam, predicted the masses of the gauge bosons W^\pm and Z^0 to be about 80 GeV and 90 GeV, respectively. In 1983, physicists at CERN [39] led by Carlo Rubia were able to produce and measure the masses of the W^\pm and Z^0 which were in complete agreement with the predictions of EW theory. The discovery of these particles may be considered as the first experimental evidence of the SSB. In the minimum formulation of the SM, a complex scalar doublet (the Higgs field) is required, which is denoted by ϕ , so that by interacting with the gauge bosons, it produces the desired breaking

$$G_{SM} \rightarrow SU(3)_C \otimes U(1)_Q, \quad (2.32)$$

where $U(1)_Q$ is a subgroup of $SU(2)_L \otimes U(1)_Y$. This breaking of symmetry occurs due to the non-zero vacuum expectation values (VEV) of the scalar field ϕ of the form

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \quad (2.33)$$

The reason for why this type of breaking occurs is as follows. The W^\pm and Z^0 gauge bosons are massive. Therefore, $SU(2)_L \otimes U(1)_Y$ can not be a symmetry of the vacuum, whereas the photon, being massless, reflects that $U(1)_Q$ is a good symmetry of the vacuum.

Now, let us write the most general renormalizable EW SM Lagrangian. It can be divided into five parts:

$$\mathcal{L}_{SM} = \mathcal{L}_{kinetic}^f + \mathcal{L}_{kinetic}^H + \mathcal{L}_{kinetic}^G + \mathcal{L}_{pot}^H + \mathcal{L}^Y. \quad (2.34)$$

The $\mathcal{L}_{kinetic}^f$ term corresponds to the fermionic sector of the SM Lagrangian. It includes both the left-handed and right-handed chiralities and can be presented as

$$\mathcal{L}_{kinetic}^f = \sum_{\psi_L} \bar{\psi}_L i \gamma^\mu D_\mu \psi_L + \sum_{\psi_R} \bar{\psi}_R i \gamma^\mu D_\mu \psi_R, \quad (2.35)$$

where $\psi_L = \frac{1}{2}(1 - \gamma_5)\psi$ stands for the left-handed weak isodoublets and $\psi_R = \frac{1}{2}(1 + \gamma_5)\psi$ for the right-handed weak isosinglets. The normal derivative, ∂_μ , is replaced by the covariant derivative, D_μ :

$$D_\mu = \partial_\mu + ig W_\mu^i \tau_i + i \frac{g'}{2} B_\mu Y, \quad (2.36)$$

to preserve the gauge invariance. Here, g and g' are the coupling constants associated with the groups $SU(2)_L$ and $U(1)_Y$ gauge groups, respectively. The corresponding generators to each gauge group are τ_i and Y in order. Moreover, W_μ^i are the three weak interaction bosons and B_μ is the single hypercharge boson. Here, the gauge boson fields W_μ^1 , W_μ^2 , W_μ^3 couple to weak isospin and B_μ couple to weak hypercharge.

The second part of the SM Lagrangian is the kinetic term for the scalar Higgs field, ϕ and is responsible for the interaction of the gauge and Higgs fields. It can be written as

$$\mathcal{L}_{kinetic}^H = (D_\mu \phi)^\dagger (D_\mu \phi), \quad (2.37)$$

with D_μ is defined in the eq. 2.36.

The corresponding kinetic term for the gauge fields reads

$$\mathcal{L}_{kinetic}^G = -\frac{1}{4} \sum_{i=1}^3 F_i^{\mu\nu} F_{\mu\nu}^i - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}, \quad (2.38)$$

where $F_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i - g\epsilon^{ijk}W_\mu^j W_\nu^k$ is the antisymmetric field strength tensor of the group $SU(2)_L$ with ϵ^{ijk} being the group structure constant and $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is that of the group $U(1)_Y$. After a proper normalization of the gauge fields, the photon, the neutral weak boson, Z^0 and the charged weak boson W_μ^\pm fields are obtained as

$$\begin{aligned} A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu, \\ Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \\ W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2), \end{aligned} \tag{2.39}$$

where

$$\sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}} \quad ; \quad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \tag{2.40}$$

with θ_W being the weak mixing angle. Finally, the photon becomes massless and the mass eigenstates for W^\pm and Z^0 bosons are obtained as

$$M_{W^\pm} = \frac{gv}{2} \quad ; \quad M_{Z^0} = \frac{\sqrt{g^2 + g'^2}}{2}. \tag{2.41}$$

The Higgs potential denoted by \mathcal{L}_H^{SM} in the SM Lagrangian reads

$$\mathcal{L}_{pot}^H = \mu^2(\phi^\dagger\phi) + \lambda(\phi^\dagger\phi)^2, \tag{2.42}$$

where μ^2 and λ are the free parameters. For $\mu^2 < 0$ the scalar field ϕ develops a non-zero vacuum expectation value at $|\phi|_{vac} = v/\sqrt{2}$ with $v = \sqrt{-\mu^2/\lambda}$. Thus, the Lagrangian gains a set of ground states for negative values of μ^2 . As a result, symmetry of the Lagrangian is spontaneously broken. In addition, through the Higgs mechanism, the Higgs mass is yielded to be equal to $m_H = \sqrt{2\lambda}v$. Notice that, all the terms explained up to now in the SM Lagrangian are CP invariant.

The final piece of the SM Lagrangian, \mathcal{L}_Y^{SM} called the Yukawa Lagrangian describes the interaction among the fermions and the Higgs field. The general

form can be expressed as²

$$\mathcal{L}^Y = \eta_{ij}^D \bar{Q}_{i,L} \phi D_{j,R} + \eta_{ij}^U \bar{Q}_{i,L} \tilde{\phi} U_{j,R} + \eta_{ij}^E \bar{l}_{i,L} \phi E_{j,R} + h.c., \quad (2.43)$$

where $\tilde{\phi} = i\tau_2 \phi^*$ and $\eta_{ij}^{U,D,E}$'s are responsible for the masses of up-down quarks and leptons, respectively. In addition, $Q_{i,L}$, $U_{j,R}$ and $D_{j,R}$ denote the left handed doublet, right handed up and right handed down quarks, respectively. Similarly, $l_{i,L}$ represent the left handed leptons and $E_{i,R}$ the right handed ones. These fermions are presented in a more elegant way in the following table:

Table 2.2: The known fermions.

Generation	Quarks		Leptons	
	Charge 2/3	Charge -1/3	Charge -1	Charge 0
	Color (R G B)	Color (R G B)	Colorless	Colorless
First	u (up)	d (down)	e (electron)	ν_e (electron neutrino)
Mass(GeV)	$0.0015 - 0.003$	$0.003 - 0.007$	0.000511	$< 3 \times 10^{-9}$
Second	c (charmed)	s (strange)	μ (muon)	ν_μ (muon neutrino)
Mass(GeV)	1.25 ± 0.09	0.095 ± 0.025	0.106	$< 190 \times 10^{-6}$
Third	t (top)	b (beauty)	τ (tau)	ν_τ (tau neutrino)
Mass(GeV)	174.2 ± 3.3	4.2 ± 0.07	1.777	$< 18.2 \times 10^{-3}$

In this table, quark masses given correspond to the approximate rest mass energy of quarks confined in hadrons since free quarks have not been observed yet. For each quark and lepton given in the table there is a corresponding antiquark and antilepton.

As mentioned before, in the SM, the fundamental fermionic constituents of the matter are quarks and leptons. All properties of these particles are summarized in the Table 2.2. These particles are placed into SM as left-handed doublets and

²In the case of massive neutrinos, there is an additional term in the Lagrangian: $\mathcal{L}'^{SM}_Y = \eta_{ij}^\nu \bar{l}_{i,L} \tilde{\phi} \nu_{j,R} + h.c.$

right-handed singlets. The left handed doublets for quarks are given by

$$\begin{pmatrix} u \\ d \end{pmatrix}_L ; \quad \begin{pmatrix} c \\ s \end{pmatrix}_L ; \quad \begin{pmatrix} t \\ b \end{pmatrix}_L , \quad (2.44)$$

and the right handed singlets for quarks are

$$d_R ; \quad u_R ; \quad s_R ; \quad c_R ; \quad b_R ; \quad t_R. \quad (2.45)$$

On the other hand, the lepton doublets are

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L ; \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L ; \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L , \quad (2.46)$$

and the right handed singlets for leptons are

$$e_R ; \quad \mu_R ; \quad \tau_R. \quad (2.47)$$

In the charged weak interactions of leptons, the coupling of W^\pm takes place strictly within a particular generation in the case of massless neutrinos . In other words, upper members of left handed lepton doublets couple to the lower members in the same doublet. That is, only the vertices $e^-\nu_e W^-$, $\mu^-\nu_\mu W^-$, and $\tau^-\nu_\tau W^-$ appear , however, there is no cross generational vertices such as $e^-\nu_\mu W^-$. The coupling of W^\pm to quarks is not quite so simple since there exist cross generational vertices as well, such as $\bar{s}u W^-$. The idea is that, the quark generations are rotated for the purposes of weak interactions such that

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L ; \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L ; \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L , \quad (2.48)$$

where d' , s' , and b' , the linear combinations of the d , s , and b , are obtained by

using the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix V_{ij}

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad (2.49)$$

where the off-diagonal elements of the CKM matrix allow flavor transitions between different generations. The experimentally measured values of the matrix elements are [40]

$$V_{ij} = \begin{pmatrix} 0.97377 \pm 0.00027 & 0.2257 \pm 0.0021 & 0.00431 \pm 0.0003 \\ 0.230 \pm 0.011 & 0.957 \pm 0.017 \pm 0.093 & 0.0416 \pm 0.0006 \\ 0.0074 \pm 0.0008 & 0.0406 \pm 0.0027 & > 0.78 \end{pmatrix}. \quad (2.50)$$

CHAPTER 3

BEYOND THE STANDARD MODEL

The SM has been extremely successful in describing the behavior of all known particles in elementary particle physics up to the EW energy scale at the order of 10^3 GeV. However, behind this energy scale, it possesses some conceptual problems which motivates us to look physics beyond the SM. There are various alternative extended models proposed for solving these problems as indicated in the introduction part. MHDM [6] is one of them. In this chapter, we will introduce the simplest extension of the SM, the so called the 2HDM [7, 8, 9].

3.1 The Two Higgs Doublet Model

Let us first present the motivation for examining the 2HDM;

- In the SM, it is assumed that the Higgs sector must be minimal having only one physical neutral Higgs scalar. However, there is no fundamental reason favoring this minimal choice. The 2HDM, being the simple extension of the SM, possesses five physical Higgs bosons, namely, a charged pair (H^\pm), two neutral CP even scalars (H^0 and h^0), and a neutral CP odd scalar (A^0).
- The ratio between the masses of top and bottom quarks is $m_t/m_b \approx 174/5 \approx 35$. According to the SM, both quarks gain mass through interactions with the same Higgs doublet. Then, we end up with an unnatural hierarchy between the corresponding Yukawa couplings. In the scope of the 2HDM, there is a possibility that the bottom receives its mass from one doublet

(say ϕ_1) and the top from the another one (say ϕ_2). Then the hierarchy of their Yukawa couplings would be more natural.

- In the framework of the SM the flavor is conserved in the lepton sector for massless neutrinos. The LFV interactions, carried by the FCNCs, exist in the extended SM, the so called ν SM, at least at one loop level, which is constructed by taking the neutrinos massive and permitting the lepton mixing mechanism [41, 42]. However, even in the ν SM, due to the smallness of the neutrino masses, the theoretical predictions of the BRs of the LFV interactions are too small to reach the experimental limits. In addition, in the quark sector the FCNCs are prohibited at tree level despite they seem not to violate any fundamental law of nature. In that aspect, the 2HDM is an elegant framework to open up the possibility of the tree level FCNCs in both lepton and quark sectors which is driven by the new scalar Higgs bosons S , the CP even scalar h^0 , and the CP odd scalar A^0 , and controlled by the Yukawa couplings.
- The 2HDM is a minimal extension in that it adds the fewest new arbitrary parameters. Instead of one free parameter of the SM, this model has six free parameters: the four Higgs masses, the ratio of the VEVs, $\tan\beta$, and a Higgs mixing angle, α . Notice that $v_1^2 + v_2^2$ is fixed by the W mass $m_W = g^2 \frac{(v_1^2 + v_2^2)}{2}$.

Having stated our motivations for an additional scalar doublet, our next task is to introduce the 2HDM. In the 2HDM, a second Higgs doublet having the same quantum numbers as the first one is introduced such that,

$$\Phi_1 = \begin{pmatrix} \phi_1^+ \\ \phi_1^0 \end{pmatrix} \quad ; \quad \Phi_2 = \begin{pmatrix} \phi_2^+ \\ \phi_2^0 \end{pmatrix}, \quad (3.1)$$

with hypercharges $Y = 1$. Parameterizing the doublets in a more convenient way we can write them in the following form

$$\Phi_1 = \begin{pmatrix} \chi^+ \\ \frac{v_1 + H^0 + i\chi^0}{\sqrt{2}} \end{pmatrix} \quad ; \quad \Phi_2 = \begin{pmatrix} H^+ \\ \frac{v_2 + H^1 + iH^2}{\sqrt{2}} \end{pmatrix}, \quad (3.2)$$

with the VEVs

$$\langle \Phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \quad ; \quad \langle \Phi_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \quad (3.3)$$

where $v = (v_1^2 + v_2^2)^{1/2} = (\sqrt{2}G_F)^{-1/2} = 246$ GeV. Here, H^0 and H^1 are the CP even, H^2 is the CP odd neutral Higgs bosons, and H^+ is the charged Higgs boson.

In the 2HDM, the Higgs part of the SM Lagrangian should be extended to include the interaction with the second Higgs doublet. Then, the kinetic term in eq. 2.37 becomes

$$(D_\mu \Phi_1)^\dagger (D^\mu \Phi_1) + (D_\mu \Phi_2)^\dagger (D^\mu \Phi_2), \quad (3.4)$$

and the most general renormalizable CP invariant Higgs potential is written in the form

$$\begin{aligned} V(\Phi_1, \Phi_2) = & \lambda_1(\Phi_1^\dagger \Phi_1 - v_1^2)^2 + \lambda_2(\Phi_2^\dagger \Phi_2 - v_2^2)^2 \\ & + \lambda_3[(\Phi_1^\dagger \Phi_1 - v_1^2) + (\Phi_2^\dagger \Phi_2 - v_2^2)]^2 \\ & + \lambda_4[(\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) - (\Phi_1^\dagger \Phi_2)(\Phi_2^\dagger \Phi_1)] \\ & + \lambda_5[Re(\Phi_1^\dagger \Phi_2) - v_1 v_2]^2 \\ & + \lambda_6[Im(\Phi_1^\dagger \Phi_2)]^2, \end{aligned} \quad (3.5)$$

where the parameters λ_i are real.

Then, what remains is the Yukawa piece of the SM Lagrangian in the presence of the two scalar doublets which is written as follows:

$$\begin{aligned} \mathcal{L}_{2HDM}^Y = & \eta_{ij}^U \bar{Q}_{i,L} \tilde{\Phi}_1 U_{j,R} + \eta_{ij}^D \bar{Q}_{i,L} \Phi_1 D_{j,R} + \xi_{ij}^U \bar{Q}_{i,L} \tilde{\Phi}_2 U_{j,R} + \xi_{ij}^D \bar{Q}_{i,L} \Phi_2 D_{j,R} \\ & + \eta_{ij}^E \bar{l}_{i,L} \Phi_1 E_{j,R} + \xi_{ij}^E \bar{l}_{i,L} \Phi_2 E_{j,R} + h.c., \end{aligned} \quad (3.6)$$

where Φ_i , for $i = 1, 2$ are the two scalar Higgs doublets, $\tilde{\Phi}_i = i\sigma_2\Phi_i$, $\eta_{ij}^{U,D,E}$ and $\xi_{ij}^{U,D,E}$ are off diagonal 3×3 matrices of the Yukawa couplings where i, j denote family indices (see Chapter 2 for the definitions of terms appearing in the Lagrangian).

As stated before, FCNCs with massless neutrino are naturally suppressed in the tree level in the SM. To avoid FCNCs at tree level, one can explicitly impose the following *ad hoc* discrete symmetry sets

$$\begin{aligned} (I) \quad & \Phi_1 \rightarrow -\Phi_1, \quad \Phi_2 \rightarrow \Phi_2, \quad D_{j,R} \rightarrow -D_{j,R}, \quad U_{j,R} \rightarrow -U_{j,R}, \\ (II) \quad & \Phi_1 \rightarrow -\Phi_1, \quad \Phi_2 \rightarrow \Phi_2, \quad D_{j,R} \rightarrow -D_{j,R}, \quad U_{j,R} \rightarrow +U_{j,R}, \end{aligned} \quad (3.7)$$

into the Yukawa Lagrangian, \mathcal{L}_{2HDM}^Y . Imposing these discrete symmetry sets, the so called model I and model II are obtained depending on whether the up-type and the down-type quarks are coupled to the same or two different scalar doublets, respectively. If no discrete symmetry is implemented into the \mathcal{L}_{2HDM}^Y , both up-type and down-type quarks and also leptons will have flavor changing (FC) couplings. This type of 2HDM is called as model III where we should take into account all the terms in the Yukawa Lagrangian given in eq. 3.6.

It is possible to make a rotation of the doublets in such a way that only one of the doublets acquire VEV so that

$$\langle \Phi_1 \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \quad ; \quad \langle \Phi_2 \rangle = 0. \quad (3.8)$$

The two doublets in this case arise of the form

$$\Phi_1 = \begin{pmatrix} \chi^+ \\ \frac{v+H^0+i\chi^0}{\sqrt{2}} \end{pmatrix} \quad ; \quad \Phi_2 = \begin{pmatrix} H^+ \\ \frac{H^1+iH^2}{\sqrt{2}} \end{pmatrix}. \quad (3.9)$$

Here H^0 and H^1 are not the neutral mass eigenstates. The neutral mass eigenstates are obtained from (H^0, H^1, H^2) as follows

$$\begin{aligned}\overline{H}^0 &= [(H^0 - v)\cos\alpha - H^1\sin\alpha], \\ h^0 &= [-(H^0 - v)\sin\alpha + H^1\cos\alpha], \\ A^0 &= H^2,\end{aligned}\tag{3.10}$$

where α is the mixing angle. It is also possible to express H^0 , H^1 and H^2 as functions of mass eigenstates

$$\begin{aligned}H^0 &= (\overline{H}^0\cos\alpha - h^0\sin\alpha) + v, \\ H^1 &= (h^0\cos\alpha + \overline{H}^0\sin\alpha), \\ H^2 &= A^0.\end{aligned}\tag{3.11}$$

Choosing $\alpha = 0$, H_1 becomes the well known mass eigenstate h_0 . As mentioned before, the model III version of the 2HDM opens up the possibility of FCNCs at the tree level and the FC part of the Yukawa Lagrangian reads

$$\mathcal{L}_{FC}^{III,Y} = \xi_{ij}^U \overline{Q}_{i,L} \tilde{\Phi}_2 U_{j,R} + \xi_{ij}^D \overline{Q}_{i,L} \Phi_2 D_{j,R} + \xi_{ij}^E \overline{l}_{i,L} \Phi_2 E_{j,R} + h.c.\tag{3.12}$$

There is another version of the 2HDM, the so called Model IV such that ϕ_1 couples and give masses to up-type quarks and ϕ_2 couples and gives masses to the down-type quarks and the VEV of the doublets are chosen

$$\langle \Phi_1 \rangle = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \quad ; \quad \langle \Phi_2 \rangle = \begin{pmatrix} 0 \\ v_2 e^{i\xi} \end{pmatrix}.\tag{3.13}$$

In this case the term containing λ_6 is replaced by $\lambda_6(Im(\phi_1^+ \phi_2) - v_1 v_2 \sin \xi)^2$ and is responsible for the CP violation in the Higgs sector.

CHAPTER 4

EXTRA DIMENSIONS

In recent years, models with extra dimensions, not yet experienced and not yet entirely understood, have been studied extensively in the literature (see for example [18]-[31]). The strong motivation to study these scenarios comes from the fact that they resolve some of the most basic mysteries of our universe such as the hierarchy problem between the two fundamental energy scales, the EW scale ($m_{EW} \sim 10^3$ GeV) and the Planck scale ($M_{Pl} \sim 10^{19}$ GeV, where the strength of gravity becomes comparable to the one of other gauge interactions. In fact, there is an important difference between these two energy scales. While the electroweak interactions have been probed at distances $\sim m_{EW}^{-1}$, gravitational interactions has not remotely been probed at distances $\sim M_{Pl}^{-1}$: it has only been accurately measured in the 1 cm range. Apart from the hierarchy problem, the cosmological constant problem (for reviews see [43, 44]), the puzzle of why the vacuum energy is driven to be a very small number, has also been tried to be solved within the extra dimensional scenarios. One possible explanation for the smallness of the cosmological constant problem can be stated as, if there is a 4D theory with only 4D sources, these will necessarily lead to an expanding universe. However, if there is 4D sources in 5D, the effects of brane sources can be balanced by a 5D cosmological constant to get a theory where the effective 4D cosmological constant would be vanishing. In this way, for an observer on a brane, the universe will still appear to be static and flat. The 5D background, however, will be curved since there exists a bulk cosmological constant in there. Such extra dimensional theories are called as warped extra dimensions, where the brane is kept flat and the extra dimensions are curved. This solution to the cosmological constant problem first pointed out by Rubakov and Shaposhnikov [45]. Finally, the extra dimensional scenario, named as the split fermion scenario [27, 28, 29],

provides an alternative view for the fermion mass hierarchies by assuming that the fermions were located at different points in the extra dimensions with the exponentially small overlaps of their wavefunctions.

Among the models with extra dimensions, emergence of the ordinary four dimensional SM as the low energy effective theory of more fundamental theory lying in higher dimensions found acceptance. The process of passing from a fundamental theory to the effective theory includes the compactification of the extra dimension(s). This compactification leads to the appearance of towers of heavy KK [32] modes of particles in $4D$ effective theory.

4.1 Large Extra Dimensions

In 1998, Nima Arkani-Hamed, Savas Dimopoulos and Gia Dvali [20, 21] tried to explain the hierarchy problem between the scales m_{EW} and M_{Pl} by assuming the existence of n extra compact spatial dimensions of large size. According to this model (called as ADD Model), the SM fields are confined to the 4D brane while gravity is free to propagate in large extra dimensions. In other words, the gravitational field has extra components in n large extra dimensions and this extra components cause it to be weaker than the other forces at long distances because it would have been diluted by the large volume of the extra dimension. In this scenario, since the SM particles are confined to a 4D brane, everything that does not involve gravity would look exactly the same as it would without the extra dimensions, even if the extra dimensions were extremely large. These extra dimensions, being compact, lead to the appearance of towers of heavy KK modes. However, since the SM particles, which are confined to a brane, would not have KK partners. The only particle in this model that must have the KK partner is the graviton. However, the graviton's KK partners interact far more weakly than the SM KK partners (see for example [47] and the references therein). Therefore, the KK partners of graviton would be much harder to be observed. One question ADD wanted to address with their set up is how large the size of the extra dimensions can be without getting into conflict with observations made up to date. To answer this question we need to match the 4D effective theory to the

fundamental higher dimensional one. Let us assume that the higher dimensional action takes the same form with Einstein-Hilbert action:

$$S_{4+n} \sim \int d^{4+n}x \sqrt{g^{(4+n)}} R^{(4+n)}. \quad (4.1)$$

Here $\sqrt{g^{(n+4)}}$ and $R^{(n+4)}$ are the metric¹ and curvature scalars in $4 + n$ dimensions. The action should be dimensionless and we need to multiply the higher dimensional Lagrangian by the appropriate power of the fundamental Planck scale (M_*), to make the action dimensionless. Since terms $d^{4+n}x$, $\sqrt{g^{(4+n)}}$ and $R^{(4+n)}$ carry dimensions of $-n - 4$, 0 and 2 , respectively, the power of M_* has to be $n + 2$. Thus, we write the higher dimensional action as

$$S_{4+n} = -M_*^{n+2} \int d^{4+n}x \sqrt{g^{(4+n)}} R^{(4+n)}. \quad (4.2)$$

Now, we will try to find how the usual 4D action

$$S_4 = -M_{Pl}^2 \int d^4x \sqrt{g^{(4)}} R^{(4)}, \quad (4.3)$$

is contained in the higher dimensional expression. For time being, it is assumed that the space-time is flat, and the extra dimensions are compact. So the metric is written as

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu - r^2 d\Omega_{(n)}^2, \quad (4.4)$$

where x_μ is a four dimensional coordinate with $\mu = 0, 1, 2, 3$, $d\Omega_{(n)}^2$ is the line element of the flat extra dimensional space, r corresponds to the radius of the extra dimension, $\eta_{\mu\nu}$ is the flat 4D metric and $h_{\mu\nu}$ is the 4D fluctuation of the metric around its minimum. Since our goal is to find out how a 4D theory emerges from a higher dimensional one, we put only 4D fluctuation into the metric. Then,

¹Metric is a quantity that establishes the measurement scale that determines the physical distances and the angles. A metric on 3D space can take the form $ds^2 = a_x dx^2 + a_y dy^2 + a_z dz^2$, where x, y, z are the three coordinates of space, and a_x, a_y, a_z can be numbers or functions of x, y, z . If $a_x = a_y = a_z = 1$, we have flat space. More complicated metrics can have cross terms, such as $dx dy$. In that case, the metric must be described by a tensor with two indices that is denoted by the coefficients a_{ij} which is the coefficient in front of $dx_i dx_j$.

the expressions for $\sqrt{g^{(n+4)}}$ and $R^{(n+4)}$ are obtained as

$$\sqrt{g^{(4+n)}} = r^n \sqrt{g^{(4)}} \quad ; \quad R^{(4+n)} = R^4. \quad (4.5)$$

Substituting these expressions into the eq. 4.2 we get

$$S_{4+n} = -M_*^{n+2} \int d\Omega_{(n)} r^n \int d^4x \sqrt{g^{(4)}} R^{(4)}, \quad (4.6)$$

where the factor $\int d\Omega_{(n)} r^n$ is simply the volume of the extra dimensional space, $V_{(n)}$. Comparing the eqs. 4.3 and 4.6, we obtain the matching relation for the gravitational couplings as

$$M_{Pl}^2 = M_*^{n+2} V_{(n)} \sim r^n M_*^{n+2}. \quad (4.7)$$

Now, we will find the matching relation for gauge couplings. Based on the assumption that every field propagates in all dimensions we can write the action as

$$S^{(4+n)} = - \int d^{4+n}x \frac{1}{4g_*^2} F_{MN} F^{MN} \sqrt{g^{(4+n)}}. \quad (4.8)$$

Here, F_{MN} with $M, N = 0, 1, \dots, 3+n$ is the higher dimensional field strength tensor and g_* is the fundamental gauge coupling. Taking the integral over the extra dimension we get

$$S^{(4)} = - \int d^4x \frac{V_{(n)}}{4g_*^2} F_{\mu\nu} F^{\mu\nu} \sqrt{g^{(4)}}. \quad (4.9)$$

Then, we obtain the matching of gauge couplings as

$$\frac{1}{g_{eff}^2} = \frac{V_n}{g_*^2}. \quad (4.10)$$

Notice that, in the eq. 4.8, $d^{4+n}x$, $\sqrt{g^{(4+n)}}$ and F_{MN} carry dimensions $-n-4$, 0 and 2, respectively. Thus, the mass dimension of g_* has to be $-n/2$ so that the action remains dimensionless. Looking at the mass dimensions of g_* and M_* we

can write

$$g_* \sim \frac{1}{M_*^{\frac{n}{2}}}. \quad (4.11)$$

Substituting this into the eq. 4.10 we get

$$\frac{1}{g_{eff}^2} = V_n M_*^n \sim r^n M_*^n. \quad (4.12)$$

Using the eq. 4.7 we obtain an equation for M_* as

$$M_* \sim \frac{M_{Pl}^{\frac{2}{n+2}}}{r^{\frac{n}{n+2}}}. \quad (4.13)$$

If we substitute this into the eq. 4.12 we get

$$\frac{1}{g_{eff}^2} \sim \frac{M_{Pl}^{\frac{2n}{n+2}}}{r^{-\frac{2n}{n+2}}}. \quad (4.14)$$

Then, r becomes

$$r \sim \frac{1}{M_{Pl}} g_{eff}^{\frac{n}{n+2}}. \quad (4.15)$$

Since $r \sim 1/M_{Pl}$ there would be no hope of finding out about the existence of these tiny extra dimensions in the near future.

Up to now, we have assumed that every field propagate in all dimensions. Now, we will deal with the restrictions on the size of extra dimensions when the SM particles are localized to 4D brane while the unobserved fields such as gravity are to propagate in extra dimensions. However, in distances as small as the size of extra dimensions, it is impossible to test gravity. Because, in such short distances electromagnetic and weak forces become more dominate than the gravitational force and the gravitational interactions have been tested at the distances of the order of millimeter. Therefore, the real bound on the size of extra dimensions becomes

$$r \leq 0.1mm, \quad (4.16)$$

if only gravity propagates in the extra dimension. From the relation $M_{Pl}^2 \sim M_*^{n+2} r^n$ we know that if we increase the value of the radius r , the fundamental

Planck scale M_* decreases. If $M_* < 1TeV$, we would have observed some effect of quantum gravity in the collider experiments. Thus, one has to impose the lowest possible value to be $M_* \sim 1TeV$. Therefore, one can say that being equal to the fundamental Planck scale, m_{EW} is the only fundamental short distance scale in nature and M_{Pl} is valid in a 4D scenario is an effective scale derived from m_{EW} . Such models are called as Large extra dimensions, proposed by Arkani- Hamed, Dimopoulos and Dvali.

Let us check, how large a radius one would need for the lowest possible value of M_* , using the eq. 4.7 the value of radius would be

$$\frac{1}{r} = M_* \left(\frac{M_*}{M_{Pl}} \right)^{\frac{2}{n}} = (1TeV) 10^{-\frac{32}{n}}. \quad (4.17)$$

Using

$$1GeV^{-1} = 2.10^{-14}cm, \quad (4.18)$$

we get

$$r \sim 2.10^{-17} 10^{-\frac{32}{n}} cm. \quad (4.19)$$

For $n = 1$, the value of $r = 2.10^{15}$ cm. Since this value is larger then the astronomical unit of 1.5×10^{13} cm, we can conclude that there cannot be one flat large extra dimension. If there are two extra dimensions, $r \sim 2$ mm. This is just a borderline for the currently probed gravitational experiments. For $n > 2$, $r < 10^{-6}$ cm. This value is so small to be measured in the near future. Then, the hierarchy problem between the fundamental Planck scale and the scale of weak interactions would have been solved so that gravity would be weaker than the other forces at long distances because it would have been diluted by the large volume of the extra dimension.

4.2 Universal and Non-Universal Extra Dimensions

As mentioned above, the compactification of the extra dimensions to a circle S_1 with a small radius R makes them imperceptibly small. But can an extra dimensional universe hide its nature so completely that none of its features distinguishes it from a 4D world? That would be hard to believe. If there are extra dimensions, fingerprints of them sure to exist. Such fingerprints are called as KK [32] particles. These new particles originate in extra dimensions, but appear to us as extra particles in our 4D space-time. In other words, they are manifestations of particles, which are in higher dimensions, in 4D such that every particle that travel in higher dimensional space is replaced by KK particles in our 4D space. A universe with extra dimensions contains both familiar particles and their KK relatives that carry extra dimensional momentum. However, a 4D dimensional space-time does not include information about higher dimensional position or momentum. This extra dimensional momentum would be seen in our 4D world as mass. Thus, KK particles should be like the ones we know (having the same charge), but heavier. If the universe contains additional dimensions, these heavier KK particles will be the first real evidence of them.

The wavefunction of a KK particle is written as Fourier decomposition of the higher dimensional wavefunction. To be concrete let us imagine a space with only one additional spatial dimension which is compactified on S^1/Z_2 orbifold ($Z_2 : y \rightarrow -y$). Then, the KK decompositions of the five dimensional (5D) fields

can be written as follows,

$$\begin{aligned}
\phi(x, y) &= \frac{1}{\sqrt{\pi R}} \phi^{(0)}(x) + \frac{\sqrt{2}}{\sqrt{\pi R}} \sum_{n=1}^{\infty} \phi^{(n)}(x) \cos\left(\frac{ny}{R}\right), \\
A_\mu(x, y) &= \frac{1}{\sqrt{\pi R}} A_\mu^{(0)}(x) + \frac{\sqrt{2}}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_\mu^{(n)}(x) \cos\left(\frac{ny}{R}\right), \\
A_5(x, y) &= \frac{\sqrt{2}}{\sqrt{\pi R}} \sum_{n=1}^{\infty} A_5^{(n)}(x) \sin\left(\frac{ny}{R}\right), \\
Q(x, y) &= \frac{1}{\sqrt{\pi R}} Q_L^{(0)}(x) + \frac{\sqrt{2}}{\sqrt{\pi R}} \sum_{n=1}^{\infty} [Q_L^{(n)}(x) \cos\left(\frac{ny}{R}\right) + Q_R^{(n)}(x) \sin\left(\frac{ny}{R}\right)], \\
U(x, y) &= \frac{1}{\sqrt{\pi R}} U_R^{(0)}(x) + \frac{\sqrt{2}}{\sqrt{\pi R}} \sum_{n=1}^{\infty} [U_R^{(n)}(x) \cos\left(\frac{ny}{R}\right) + U_L^{(n)}(x) \sin\left(\frac{ny}{R}\right)], \\
D(x, y) &= \frac{1}{\sqrt{\pi R}} D_R^{(0)}(x) + \frac{\sqrt{2}}{\sqrt{\pi R}} \sum_{n=1}^{\infty} [D_R^{(n)}(x) \cos\left(\frac{ny}{R}\right) + U_L^{(n)}(x) \sin\left(\frac{ny}{R}\right)].
\end{aligned} \tag{4.20}$$

Fields even under the Z_2 symmetry will have zero modes and they correspond to the SM particles in our usual 4D world whereas fields odd under Z_2 symmetry will only have KK modes and will be absent in the low energy spectrum so that, in the four dimensional Lagrangian, we get rid of the the zero modes of wrong chirality (*i.e.*, Q_R , U_L , and D_L) and the fifth component of the gauge field, A_5 . If we could measure and study their properties, they would tell us everything about the higher dimensional space.

The electroweak, $SU(2)_L \times U(1)_Y$, part of the SM Lagrangian in 5D can be written as follows

$$\mathcal{L} = \int_0^{\pi R} dy (\mathcal{L}_{kinetic}^f + \mathcal{L}^H + \mathcal{L}_{kinetic}^G + \mathcal{L}^Y), \tag{4.21}$$

where $y \equiv x^4$ is the coordinate along the extra dimension. The fermionic piece of the 5D Lagrangian is defined as

$$\mathcal{L}_{kinetic}^f = \bar{Q}(i\Gamma^M D_M)Q + \bar{U}(i\Gamma^M D_M)U + \bar{D}(i\Gamma^M D_M)D, \tag{4.22}$$

where, $M, N = 0, 1, 2, 3, 4$ corresponds to the five-dimensional Lorentz indices.

The fermionic fields Q , U and D denotes five dimensional generic quark doublet, up type quark singlet, and down type quark singlet, respectively. Unlike in the SM, these fermionic fields have both chiralities and are all vector type. The covariant derivative, D_M is defined as $D_M = \partial_M - i\tilde{g}W_M^a T^a - i\tilde{g}'B_M Y$ with \tilde{g} is the five dimensional gauge coupling constant of the group $SU(2)_L$ and \tilde{g}' is that of $U(1)_Y$. Here, T^a and Y are the corresponding generators. In addition, Γ_M are the five dimensional gamma matrices with $\Gamma_\mu = \gamma_\mu$ and $\Gamma_4 = i\gamma_5$. The Higgs piece of the 5D Lagrangian is

$$\mathcal{L}^H = (D_M \phi)^\dagger (D^M \phi) - V(\phi), \quad (4.23)$$

and the gauge piece is

$$\mathcal{L}_{kinetic}^G = -\frac{1}{4} \sum_{i=1}^3 F_i^{MN} F_{MN}^i - \frac{1}{4} F^{MN} F_{MN}. \quad (4.24)$$

The field strength tensors associated with the $SU(2)_L$ and $U(1)_Y$ gauge group can be expressed as

$$F_{MN}^i = \partial_M W_N^i - \partial_N W_M^i + g\epsilon^{ijk} W_M^j W_N^k, \quad (4.25)$$

and

$$F_{MN} = \partial_M B_N - \partial_N B_M, \quad (4.26)$$

respectively. Finally the Yukawa piece reads

$$\mathcal{L}^Y = \bar{Q} \tilde{Y}_u \phi^c U + \bar{Q} \tilde{Y}_d \phi D. \quad (4.27)$$

In the above equation, the fields ϕ and $\phi^c = i\tau^2 \phi^*$ stand for the standard Higgs doublet and its charge conjugated field as each refers to each in order. Finally, \tilde{Y}_u and \tilde{Y}_d correspond the Yukawa matrices in the five dimensional theory which are responsible for mixing different generations. For simplicity lepton or gluon indices are not included.

Substituting the KK decompositions of the five dimensional fields given above

into the eq. 4.21 and integrating over the extra dimension y , we obtain the effective four dimensional Lagrangian. One can realize simply that the KK excitations receive mass not only due to the vacuum expectation value of the zero-mode Higgs but also from the kinetic energy term in the five dimensional Lagrangian. The mass of the n^{th} KK particle is given by $m_n^{KK} = \sqrt{m_0^2 + m_n^2}$. Here m_0 corresponds to the zero mode mass and $m_n = n/R$.

Depending on the underlying fundamental theory, extra dimensions may or may not be accessible to all fields in the model. According to this accessibility, extra dimensions can be grouped into two, including "universal extra dimensions" (see for example (UED) [22, 23, 24] and the references therein) and "non-universal extra dimensions" (NUED) (see for example [25, 26] and the references therein), respectively. In a theory with universal extra dimensions, all fields in the model feel the extra dimensions. Conservation of extra dimensional momentum leads to the key feature of such theories that KK number at each elementary interaction vertex is conserved. As a result of this feature, production of an isolated KK particle at colliders is forbidden. Instead they are produced in pairs. This, in turn, implies that there is no tree-level contribution to weak decays of quarks and leptons. They enter into the calculations only through loop corrections. However, in a theory with non-universal extra dimensions, some of the SM fields are confined to a 4D brane and the others live in the bulk. In this case, the Lagrangian contains localizing delta function which permits KK number violating couplings. Then, the tree-level interactions of KK modes with the ordinary particles can exist.

4.3 The Randall-Sundrum Model

The large extra dimensions which are discussed in the previous section took the advantage of the fact that branes could trap particles and force but neglected the energy that the branes themselves could carry. However, according to Einstein's theory of general relativity, gravitational field is induced by means of energy, which means that when branes carry energy, they should curve space and time. Lisa Randall and Raman Sundrum [30, 31] tried to explain how space-time

would be curved in the presence of two 4D energetic branes that bounded the extra dimension of space where the bulk² geometry is anti-de Sitter³, by solving Einstein's gravity equations based on the assumption that both the bulk and the branes have energy. In this space-time the 4D branes and any single slice along the fifth dimension are completely flat. But the 5D space-time under consideration is nonetheless curved. The technical term for this type of geometry is 'warped'. This section focusses on a warped five-dimensional world that provides an alternative approach to explain the huge discrepancy between m_{EW} and M_{Pl} without the need for a large extra dimension. In this scenario (called as Randall-Sundrum (RS1) Model), the geometry contains two 4D flat branes, the Planck brane where the gravity is localized and the TeV brane where all SM particles are confined, that bound a fifth dimension which is compactified to S^1/Z_2 orbifold. The two 4D flat branes with opposite tensions, which reside at the orbifold fixed points together with a finely tuned non-vanishing 5D cosmological constant Λ , serve as sources for 5D gravity. Since the two branes are completely flat, the induced metric at every point along the extra dimension has to be the ordinary flat 4D Minkowski metric, and the components of the 5D metric depend only on the fifth coordinate, y . Thus, the most general space-time metric satisfying these properties is given by

$$ds^2 = e^{-A(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (4.28)$$

where $e^{-A(y)}$ is called as the warp-factor which determines the amount of curvature along the extra dimension. It is also possible to write this metric in a conformally flat form where there is an overall factor. To go into the conformally flat frame, all we need to do is to make a coordinate transformation of the form $z = z(y)$ such that dy and dz are related by

$$e^{-A(z)/2} dz = dy. \quad (4.29)$$

²Full higher-dimensional space.

³Space-time with constant negative curvature.

Then, the metric in the eq. 4.28 becomes

$$ds^2 = e^{-A(z)}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2), \quad (4.30)$$

and we get

$$g_{MN} = e^{-A(z)}\tilde{g}_{MN}, \quad (4.31)$$

where \tilde{g}_{MN} is the flat metric, $\tilde{g}_{MN} = \eta_{MN}$.

4.3.1 The Einstein tensor and the brane tensions

The starting point is the action (see [49] for example)

$$S = \int d^5x \sqrt{g}(M_*^3 R), \quad (4.32)$$

where M_* is the 5D Planck scale, $R = g^{MN}R_{MN}$ and R_{MN} reads

$$R_{MN} = \Gamma_{MK,N}^K - \Gamma_{MN,K}^K - \Gamma_{MN}^K \Gamma_{KL}^L + \Gamma_{ML}^K \Gamma_{NK}^L, \quad (4.33)$$

with the connections

$$\Gamma_{MN}^K = g^{KL}\Gamma_{LMN} = \frac{1}{2}g^{KL}(g_{LM,N} + g_{LN,M} - g_{MN,L}). \quad (4.34)$$

After the variation of this action with respect to g_{MN} (see [48] and [49] for details) we get

$$\delta S = \int d^5x [\delta\sqrt{g}M_*^3 R + \sqrt{g}M_*^3 \delta g^{MN} R_{MN}], \quad (4.35)$$

where

$$\delta\sqrt{g} = \frac{1}{2}\delta g_{MN}g^{MN}\sqrt{g} \quad ; \quad \delta g^{MN} = -g^{MK}g^{NL}\delta g_{KL}. \quad (4.36)$$

Finally we obtain

$$\delta S = -M_*^3 \int d^5x (R^{MN} - \frac{1}{2}g^{MN}R)\sqrt{g}\delta g_{MN}, \quad (4.37)$$

with the Einstein tensor $G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R$. Now, we will calculate this tensor for the special metric given in eq. 4.31. From now on, since $\tilde{g}_{MN} = \eta_{MN}$, all covariant derivatives $\tilde{\nabla}_M$ (which are with respect to the metric \tilde{g}) will be replaced by the normal derivative ∂_M . Let us calculate R_{MN} term by term. The first term is:

$$\begin{aligned}
\Gamma_{MK,N}^K &= \{g^{KS}\Gamma_{SMK}\}_{,N} \\
&= \frac{1}{2}\{g^{KS}[g_{SM,K} + g_{SK,M} - g_{MK,S}]\}_{,N} \\
&= \frac{1}{2}\{e^A\eta^{KS}[(e^{-A}\eta_{SM}),_K + (e^{-A}\eta_{SK}),_M - (e^{-A}\eta_{MK}),_S]\}_{,N} \\
&= \frac{1}{2}\{\eta^{KS}[-\eta_{SM}\partial_K A - \eta_{SK}\partial_M A + \eta_{MK}\partial_S A]\}_{,N} \\
&= \frac{1}{2}\{\eta^{KS}[-\eta_{SM}\partial_N\partial_K A - \eta_{SK}\partial_N\partial_M A + \eta_{MK}\partial_N\partial_S A]\} \\
&= -\frac{1}{2}\eta_M^K\partial_N\partial_K A - \frac{1}{2}\eta_S^S\partial_N\partial_M A + \frac{1}{2}\eta_M^S\partial_N\partial_S A,
\end{aligned} \tag{4.38}$$

where

$$\eta_M^N = \delta_{MN} \quad \eta_M^M = d, \tag{4.39}$$

with the space-time dimension d . Substituting these equations into $\Gamma_{MK,N}^K$ we obtain the first term in R_{MN} as

$$\Gamma_{MK,N}^K = -\frac{d}{2}\partial_N\partial_M A. \tag{4.40}$$

By using the same procedure we get the other terms as:

$$\begin{aligned}
\Gamma_{MN,K}^K &= \{g^{KS}\Gamma_{SMN}\}_{,K} \\
&= \frac{1}{2}\{g^{KS}[g_{SM,N} + g_{SN,M} - g_{MN,S}]\}_{,K} \\
&= \frac{1}{2}\{e^A\eta^{KS}[(e^{-A}\eta_{SM}),_N + (e^{-A}\eta_{SN}),_M - (e^{-A}\eta_{MN}),_S]\}_{,K} \\
&= \frac{1}{2}\{\eta^{KS}[-\eta_{SM}\partial_N A - \eta_{SN}\partial_M A + \eta_{MN}\partial_S A]\}_{,K} \\
&= \frac{1}{2}\{\eta^{KS}[-\eta_{SM}\partial_K\partial_N A - \eta_{SN}\partial_K\partial_M A + \eta_{MN}\partial_K\partial_S A]\} \\
&= -\frac{1}{2}\eta_M^K\partial_K\partial_N A - \frac{1}{2}\eta_N^K\partial_K\partial_M A + \frac{1}{2}\eta^{KS}\eta_{MN}\partial_K\partial_S A \\
&= -\partial_M\partial_N A + \frac{1}{2}\eta_{MN}\partial^2 A,
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\Gamma_{MN}^K\Gamma_{KL}^L &= g^{KS}\Gamma_{SMN}g^{LP}\Gamma_{PKL} \\
&= \frac{1}{2}g^{KS}[g_{SM,N} + g_{SN,M} - g_{MN,S}]\frac{1}{2}g^{LP}[g_{PK,L} + g_{PL,K} - g_{KL,P}] \\
&= \frac{1}{4}e^A\eta^{KS}[(e^{-A}\eta_{SM}),_N + (e^{-A}\eta_{SN}),_M - (e^{-A}\eta_{MN}),_S] \\
&\quad \times e^A\eta^{LP}[(e^{-A}\eta_{PK}),_L + (e^{-A}\eta_{PL}),_K - (e^{-A}\eta_{KL}),_P] \\
&= \frac{1}{4}\eta^{KS}[-\eta_{SM}\partial_N A - \eta_{SN}\partial_M A + \eta_{MN}\partial_S A] \\
&\quad \times \eta^{LP}[-\eta_{PK}\partial_L A - \eta_{PL}\partial_K A + \eta_{KL}\partial_P A] \\
&= \frac{1}{4}[-\eta_M^K\partial_N A - \eta_N^K\partial_M A + \eta^{KS}\eta_{MN}\partial_S A] \\
&\quad \times [-\eta_K^L\partial_L A - \eta_L^L\partial_K A + \eta_K^P\partial_P A] \\
&= \frac{1}{4}[-\eta_M^K\partial_N A - \eta_N^K\partial_M A + \eta_{MN}\partial^K A][-d\partial_K A] \\
&= \frac{d}{2}\partial_M A\partial_N A - \frac{d}{4}\eta_{MN}(\partial A)^2,
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
\Gamma_{ML}^K \Gamma_{NK}^L &= g^{KS} \Gamma_{SML} g^{LP} \Gamma_{PNK} \\
&= \frac{1}{2} g^{KS} [g_{SM,L} + g_{SL,M} - g_{ML,S}] \frac{1}{2} g^{LP} [g_{PN,K} + g_{PK,N} - g_{NK,P}] \\
&= \frac{1}{4} e^A \eta^{KS} [(e^{-A} \eta_{SM}),_L + (e^{-A} \eta_{SL}),_M - (e^{-A} \eta_{ML}),_S] \\
&\quad \times e^A \eta^{LP} [(e^{-A} \eta_{PN}),_K + (e^{-A} \eta_{PK}),_N - (e^{-A} \eta_{NK}),_P] \\
&= \frac{1}{4} \eta^{KS} [-\eta_{SM} \partial_L A - \eta_{SL} \partial_M A + \eta_{ML} \partial_S A] \\
&\quad \times \eta^{LP} [-\eta_{PN} \partial_K A - \eta_{PK} \partial_N A - \eta_{NK} \partial_P A] \\
&= \frac{1}{4} [-\eta_M^K \partial_L A - \eta_L^K \partial_M A + \eta^{KS} \eta_{ML} \partial_S A] \\
&\quad \times [-\eta_N^L \partial_K A - \eta_K^L \partial_N A + \eta^{LP} \eta_{NK} \partial_P A] \\
&= \frac{1}{4} [-\eta_M^K \partial_L A - \eta_L^K \partial_M A + \eta_{ML} \partial^K A] \\
&\quad \times [-\eta_N^L \partial_K A - \eta_K^L \partial_N A + \eta_{NK} \partial_L A] \\
&= \frac{1}{4} (2+d) \partial_M A \partial_N A - \frac{1}{2} \eta_{MN} (\partial A)^2.
\end{aligned} \tag{4.43}$$

Finally R_{MN} becomes

$$\begin{aligned}
R_{MN} &= \frac{2-d}{2} \partial_M \partial_N A + \frac{2-d}{4} \partial_M A \partial_N A \\
&\quad + \frac{d-2}{4} \eta_{MN} (\partial A)^2 - \frac{1}{2} \eta_{MN} \partial^2 A.
\end{aligned} \tag{4.44}$$

Now, we will find R using $R = g^{MN} R_{MN}$ as follows:

$$\begin{aligned}
R &= e^A \eta^{MN} R_{MN} \\
&= e^A \left[\frac{2-d}{2} \partial^2 A + \frac{2-d}{4} (\partial A)^2 + \frac{d(d-2)}{4} (\partial A)^2 - \frac{d}{2} \partial^2 A \right] \\
&= e^A \left[(1-d) \partial^2 A + \frac{(2-d)(1-d)}{4} (\partial A)^2 \right].
\end{aligned} \tag{4.45}$$

Substituting the eqs. 4.44 and 4.45 into the Einstein tensor we get

$$\begin{aligned}
G_{MN} &= \frac{2-d}{2} \left\{ \frac{1}{2} \partial_M A \partial_N A + \partial_M \partial_N A \right. \\
&\quad \left. - \eta_{MN} [\partial^K \partial_K A - \frac{d-3}{4} \partial^K A \partial_K A] \right\}.
\end{aligned} \tag{4.46}$$

Here $d = 5$ stands for the number of dimensions. Using this expression we can evaluate the non-vanishing terms G_{55} and $G_{\mu\nu}$. Let us start with G_{55} :

$$G_{55} = -\frac{3}{2}\left\{\frac{1}{2}A'^2 + A'' - \eta_{55}[-A'' + \frac{1}{2}A'^2]\right\}, \quad (4.47)$$

where $A' = \partial_5 A$ and $\eta_{55} = -1$ and we get

$$G_{55} = -\frac{3}{2}A'^2. \quad (4.48)$$

$G_{\mu\nu}$ can be obtained in the same way:

$$G_{\mu\nu} = -\frac{3}{2}\left\{\frac{1}{2}\partial_\mu A \partial_\nu A + \partial_\mu \partial_\nu A - \eta_{\mu\nu}[\partial^K \partial_K A - \frac{1}{2}\partial^K A \partial_K A]\right\}. \quad (4.49)$$

Since $A = A(z)$, the terms including 4D differentiation vanish and we obtain,

$$G_{\mu\nu} = \frac{3}{2}\eta_{\mu\nu}(-A'' + \frac{1}{2}A'^2). \quad (4.50)$$

Now, we consider the 5D Einstein action for gravity with a bulk cosmological constant Λ :

$$S = \int d^5x \sqrt{g}(M_*^3 R - \Lambda). \quad (4.51)$$

Taking the variation of the action with respect to metric

$$\delta S = \int d^5x [\delta\sqrt{g}M_*^3 R + \sqrt{g}M_*^3 \delta g^{MN} R_{MN} - \delta\sqrt{g}\Lambda], \quad (4.52)$$

we get (see eq. 4.37 for the variation of the first term in eq. 4.51)

$$\delta S = \int d^5x [-M_*^3 G^{MN} - \frac{1}{2}\Lambda g^{MN}] \sqrt{g} \delta g_{MN}. \quad (4.53)$$

Then, we can simply write the Einstein tensor as

$$G_{MN} = -\frac{1}{2M_*} \Lambda g_{MN}. \quad (4.54)$$

The 55 component of Einstein tensor will then be:

$$\frac{3}{2}A'^2 = \frac{1}{2M_*} \Lambda g_{55} = -\frac{1}{2M_*} \Lambda e^{-A(z)}. \quad (4.55)$$

and A' reads,

$$A' = \sqrt{\frac{-\Lambda}{3M_*^3}} e^{-A(z)/2}. \quad (4.56)$$

There exists a solution if and only if $\Lambda < 0$. This means that the important case for us will be considering anti-de Sitter spaces, that is the spaces with negative cosmological constant. Now, let us take $f = e^{-A(z)/2}$ and, therefore, $f' = \frac{1}{2}A'(z)f$. Substituting them into the eq. 4.56 we obtain

$$-\frac{f'}{f^2} = \frac{1}{2} \sqrt{\frac{-\Lambda}{3M_*^3}}. \quad (4.57)$$

Solving this differential equation we get a relation for f such that,

$$f = e^{-A(z)} = \frac{1}{(kz + c_0)^2}, \quad (4.58)$$

where $k^2 = \frac{-\Lambda}{12M_*^3}$. If we choose $e^{-A(0)} = 1$,

$$e^{-A(0)} = \frac{1}{c_0^2} = 1, \quad (4.59)$$

we get $c_0 = 1$ and, finally, we obtain

$$e^{-A(z)} = \frac{1}{(kz + 1)^2}. \quad (4.60)$$

This solution must be symmetric under $z \rightarrow -z$ reflection since we are on a S^1/Z_2 orbifold and, therefore, we take

$$e^{-A(z)} = \frac{1}{(k|z| + 1)^2}. \quad (4.61)$$

The RS metric is then obtained as

$$ds^2 = \frac{1}{(k|z| + 1)^2} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2). \quad (4.62)$$

Using the eq. 4.29, we obtain $A(z)$ as follows:

$$\frac{dz}{k|z| + 1} = dy, \quad (4.63)$$

and solving for y we get

$$y = \frac{\ln(k|z| + 1)}{kz/|z|} + c_1. \quad (4.64)$$

If we choose $y = 0$ to correspond to $z = 0$ the constant c_1 becomes $c_1 = 0$. Then, we have

$$\frac{1}{(k|z| + 1)^2} = e^{-2k|y|}, \quad (4.65)$$

and the final form of the RS metric in y coordinates becomes

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (4.66)$$

where y corresponds to the physical distance along the extra dimension, since in that metric there is no warp factor in front of dy^2 term and the Planck (TeV) brane is located at $y = 0$ ($y = r_0$).

At this stage, we would like to check whether the $4D$ components of the Einstein tensor given in eq. 4.50 are also satisfied or not. Using the eq. 4.61, $A(z)$ is obtained as

$$A(z) = \ln(k|z| + 1)^2. \quad (4.67)$$

The derivative of $A(z)$ with respect to z gives

$$A'(z) = \frac{2(k|z| + 1)kz/|z|}{(k|z| + 1)^2} = \frac{2k\varepsilon(z)}{(k|z| + 1)}, \quad (4.68)$$

where $\frac{z}{|z|} = \varepsilon(z) = (\theta(z) - \theta(-z))$ and one more derivative of $A(z)$ with respect to z reads

$$A''(z) = -\frac{2k^2}{(k|z| + 1)^2} + \frac{4k}{k|z| + 1} (\delta(z) - \delta(z - z_1)). \quad (4.69)$$

Substituting A' and A'' into eq. 4.50 we get

$$G_{\mu\nu} = -\frac{3}{2}\eta_{\mu\nu}\left\{\frac{4k^2}{(k|z|+1)^2} - \frac{4k[\delta(z) - \delta(z-z_1)]}{k|z|+1}\right\}. \quad (4.70)$$

Here the first term is the contribution of the bulk cosmological constant into the energy momentum tensor. The remaining delta functions should be compensated by the additional sources onto the branes. To do this we need to find the energy-momentum tensor of a brane tension term V using the action

$$S = \int d^4x V \sqrt{g^{ind}} = \int d^5x V \frac{\sqrt{g}}{\sqrt{g^{55}}} \delta(y), \quad (4.71)$$

for a flat brane at $y = 0$. Taking the variation of this action with respect the metric $g_{\mu\nu}$ we get

$$\delta S = \int d^4x V \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \sqrt{g^{ind}}, \quad (4.72)$$

and by using the energy-momentum tensor for a brane

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}, \quad (4.73)$$

we obtain $T_{\mu\nu}$ as

$$T_{\mu\nu} = \frac{1}{2\sqrt{g^{55}}} g_{\mu\nu} V \delta(y). \quad (4.74)$$

Therefore, at the branes we need to have two brane tensions to compensate the delta functions. Thus, we need the equality

$$-\frac{3}{2}\eta_{\mu\nu}\left[-\frac{4k(\delta(z) - \delta(z-z_1))}{k|z|+1}\right] = \frac{\eta_{\mu\nu}}{2M_*^3}\left[\frac{V_0\delta(z) + V_1\delta(z-z_1)}{k|z|+1}\right]. \quad (4.75)$$

Then, one can simply conclude that, the brane tensions at the two fixed points will have opposite signs and they are given by

$$V_0 = -V_1 = 12kM_*^3. \quad (4.76)$$

Finally, using the expression $k^2 = \frac{-\Lambda}{12M_*^3}$ the bulk cosmological constant is obtained

as

$$\Lambda = -\frac{V_0^2}{12M_*^3} \quad ; \quad V_1 = -V_0. \quad (4.77)$$

4.3.2 The Radion

There is a potential gap in this scenario that needs to be filled. In the Randall-Sundrum scenario, it is assumed that the brane dynamics would naturally lead to branes to be located at a modest distance apart but it is not explicitly explained that how the distance between the two branes is established since their solution is valid for any choice of r_0 . If the distance between the two branes remains undetermined, when the energy or the temperature of the universe evolves, the branes will have the potential to move toward or against to each other. If the brane separation could change, the universe would not evolve in the way it is supposed to in 4D and thus the warped 5D universe would not agree with the cosmological observations. Goldberger and Wise (GW) [50] did the important research that closed this gap in the theory by fixing⁴ $r_0 \sim 30/k$ without introducing any large finetuning. They suggested that, in addition to the graviton, there is a massive particle that lives in the 5D bulk for which the equilibrium configuration for the field and the branes would involve a modest brane separation. Denoting the scalar field in the bulk by ϕ , the action under consideration will be (see for example [47] and [48])

$$S = \int d^5x \sqrt{g} M_*^3 R + \int d^5x \sqrt{g} \frac{1}{2} [(\nabla\phi)^2 - V(\phi)] - \int d^4x \sqrt{g_4} \lambda_P(\phi) - \int d^4x \sqrt{g_4} \lambda_T(\phi), \quad (4.78)$$

where the first term is the usual 5D Einstein-Hilbert action and the second term is the bulk action for the scalar field, while the next two terms, with $\sqrt{g_4}$ being the induced metric on the branes, are the brane induced potentials for the scalar field on the Planck and on the TeV branes. We will look for an ansatz of the

⁴In the negative tension brane where we live all mass scales are exponentially suppressed, $e^{-kr_0} M_{Pl} \sim 1$ TeV. Therefore, $kr_0 \sim \ln 10^{16} \sim 30$ since the Planck scale $M_{Pl} = 10^{16}$ TeV.

background metric again of generic form as in the RS case such that,

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2. \quad (4.79)$$

The Einstein equations will be exactly the same as we have derived for the RS case, except the energy momentum tensor that is derived from the action of the scalar field. Let us take the variation of this action with respect to metric term by term. The variation of the first term is given in eq. 4.37. The metric variation for the second term

$$\int d^5x \sqrt{g} \left[\frac{1}{2} \nabla \phi \nabla \phi - V(\phi) \right] = \int d^5x \sqrt{g} \left[\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \right], \quad (4.80)$$

reads

$$\begin{aligned} \delta \int d^5x \sqrt{g} \left[\frac{1}{2} \nabla \phi \nabla \phi - V(\phi) \right] &= \int d^5x \delta \sqrt{g} \left[\frac{1}{2} (\partial \phi)^2 - V(\phi) \right] \\ &\quad + \int d^5x \sqrt{g} \frac{1}{2} \delta g^{MN} \partial_M \phi \partial_N \phi \\ &= \int d^5x \left(\frac{1}{2} \sqrt{g} g^{KL} \delta g_{KL} \right) \left[\frac{1}{2} (\partial \phi)^2 - V(\phi) \right] \\ &\quad - \int d^5x \sqrt{g} \frac{1}{2} (g^{MK} g^{NL} \delta g_{KL}) \partial_M \phi \partial_N \phi. \end{aligned} \quad (4.81)$$

By making simplification, we obtain the variation as,

$$\begin{aligned} \delta \int d^5x \sqrt{g} \left[\frac{1}{2} \nabla \phi \nabla \phi - V(\phi) \right] &= \int d^5x \left\{ \frac{1}{2} g^{KL} \left[\frac{1}{2} (\partial \phi)^2 - V(\phi) \right] \right. \\ &\quad \left. - \frac{1}{2} \partial^K \phi \partial^L \phi \right\} \sqrt{g} \delta g_{KL}. \end{aligned} \quad (4.82)$$

Finally, we will look at the metric variation of the last two terms which can be written in a more compact form as

$$\int d^4x \sqrt{g_4} \lambda_P(\phi) + \int d^4x \sqrt{g_4} \lambda_T(\phi) = \int d^5x \frac{\sqrt{g}}{\sqrt{g_{55}}} \sum_i \lambda_i(\phi) \delta(y - y_i), \quad (4.83)$$

where i denote TeV and Planck branes. If we do the metric variation to the combination of third and the fourth terms we get

$$\begin{aligned}
\int d^4x \sqrt{g_4} \lambda_P(\phi) + \int d^4x \sqrt{g_4} \lambda_T(\phi) &= \int d^5x \delta \sqrt{g} \frac{1}{\sqrt{g_{55}}} \sum_i \lambda_i(\phi) \delta(y - y_i) \\
&= \int d^5x \left(\frac{1}{2} \delta g_{KL} g^{KL} \sqrt{g} \right) \frac{1}{\sqrt{g_{55}}} \sum_i \lambda_i(\phi) \delta(y - y_i) \\
&= \int d^5x \left[\frac{1}{2} g_\mu^K g_\nu^L g^{\mu\nu} \sum_i \lambda_i(\phi) \delta(y - y_i) \right] \sqrt{g} \delta g_{KL}.
\end{aligned} \tag{4.84}$$

Replacing K and L with M and N , respectively, we obtain T^{MN} as

$$\begin{aligned}
T^{MN} &= \frac{1}{2} g^{MN} \left[\frac{1}{2} (g_{RS} \partial^R \phi \partial^S \phi) - V(\phi) \right] \\
&\quad - \frac{1}{2} \partial^M \phi \partial^N \phi - \frac{1}{2} g_\mu^K g_\nu^L g^{\mu\nu} \sum_i \lambda_i(\phi) \delta(y - y_i).
\end{aligned} \tag{4.85}$$

By considering the equality (see Appendix B for its derivation)

$$R_{MN} = \kappa^2 \tilde{T}_{MN}, \tag{4.86}$$

where

$$\tilde{T}_{MN} = T_{MN} - \frac{1}{3} g_{MN} T, \tag{4.87}$$

we get

$$4A'^2 - A'' = -\frac{2\kappa^2}{3} V(\phi_0) - \frac{\kappa^2}{3} \sum_i \lambda_i(\phi_0) \delta(y - y_i). \tag{4.88}$$

On the other hand, the 55 component of the Einstein equation is obtained as

$$G_{55} = \frac{\kappa^2}{2} \phi_0'^2 - \kappa^2 V(\phi_0), \tag{4.89}$$

and we get

$$A'^2 = \frac{\kappa^2}{12} \phi_0'^2 - \frac{\kappa^2}{6} V(\phi_0). \tag{4.90}$$

In these equations ϕ_0 denotes the solution of the scalar field, which is assumed to be only a function of y : $\phi = \phi_0(y)$. In addition to these two equations, the bulk scalar equation of motion is found by using

$$\partial_M \frac{\partial \mathcal{L}}{\partial (\partial_M \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \quad (4.91)$$

as

$$\partial_M (\sqrt{g} g^{MN} \partial_N \phi) = -\frac{\partial V}{\partial \phi} \sqrt{g}. \quad (4.92)$$

We can write the term $\partial_M (\sqrt{g} g^{MN} \partial_N \phi)$ also in the form

$$\begin{aligned} \partial_M (\sqrt{g} g^{MN} \partial_N \phi) &= \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) + \partial_5 (\sqrt{g} g^{55} \partial_5 \phi) \\ &= \partial_\mu (e^{-4A} e^{2A} \partial_\nu \phi) + \partial_5 (e^{-4A} (-1) \partial_5 \phi) \\ &= e^{-2A} \partial_\mu \partial_\nu \phi - 2e^{-2A} \partial_\mu A \partial_\nu \phi - e^{-4A} \partial_5 \partial_5 \phi + 4e^{-4A} \partial_5 A \partial_5 \phi. \end{aligned} \quad (4.93)$$

Since $\phi = \phi_0(y)$ and $A = A(y)$ we get

$$-e^{-4A} \phi_0'' + 4e^{-4A} A' \phi_0' = -\frac{\partial V}{\partial \phi} e^{-4A}. \quad (4.94)$$

Including the brane tensions we can write

$$\phi_0'' - 4A' \phi_0' = \frac{\partial V(\phi_0)}{\partial \phi} + \sum_i \frac{\partial \lambda_i(\phi_0)}{\partial \phi} \delta(y - y_i). \quad (4.95)$$

The metric itself have to be continuous. However, there is no requirement that the derivative of the metric to be continuous. In the first Einstein equation given in 4.88 there exists the explicit delta function term $-\frac{\kappa^2}{3} \sum_{i=P,T} \lambda_i(\phi_0) \delta(y - y_i)$ at the branes. It seems not the be balanced by anything else unless there is a jump in the derivative A' at the branes. If the derivative jumps from $A'(0 - \epsilon)$ to $A'(0 + \epsilon)$, this implies that locally A' contains a term of the form

$$A'(y = 0) \sim [A'(0 + \epsilon) - A'(0 - \epsilon)] \varepsilon(y), \quad (4.96)$$

where $\varepsilon(y)$ is the unitstep function. Taking one more derivative with respect to y we get

$$A''(y=0) \sim [A'(0+\epsilon) - A'(0-\epsilon)]\delta(y). \quad (4.97)$$

Thus, the delta function is proportional to the jump of the derivative of A' . In the same way, the delta function in the bulk scalar equation of motion will be proportional to the jump of the derivative ϕ'_0 . Therefore, the boundary conditions, (or jump equations) will be given by

$$\begin{aligned} [A']_i &= \frac{\kappa^2}{3}\lambda_i(\phi_0), \\ [\phi'_0]_i &= \frac{\partial\lambda_i(\phi_0)}{\partial\phi}. \end{aligned} \quad (4.98)$$

These are coupled second order differential equations. So, they are quite hard to solve. Only for specific potentials solution can be simplified. Defining the function $W(\phi)$ such that

$$\begin{aligned} A' &\equiv \frac{\kappa^2}{6}W(\phi_0), \\ \phi'_0 &\equiv \frac{1}{2}\frac{\partial W}{\partial\phi}, \end{aligned} \quad (4.99)$$

and substituting these equations into the 55 component of Einstein equation we get

$$V(\phi) = \frac{1}{8}\left(\frac{\partial W}{\partial\phi}\right)^2 - \frac{\kappa^2}{6}W(\phi)^2. \quad (4.100)$$

This is called as the consistency equation since when we plug in the expressions for A' and ϕ'_0 into the Einstein and scalar equations, we will find that all equations are satisfied. Then, the jump equations are given by

$$\begin{aligned} \frac{1}{2}[W(\phi_0)]_i &= \lambda_i(\phi_0), \\ \frac{1}{2}\left[\frac{\partial W}{\partial\phi}\right]_i &= \frac{\partial\lambda_i(\phi_0)}{\partial\phi}. \end{aligned} \quad (4.101)$$

If W were given, the coupled second order differential equations would be reduced to first order equations that are easy to solve. We can simply start with a superpotential that will produce V with the required properties. In our case, we would like the bulk potential to include cosmological constant term (independent of ϕ) and mass term (quadratic in ϕ) in its simplest form. So we choose

$$W(\phi) = \frac{6k}{\kappa^2} - u\phi^2. \quad (4.102)$$

where the first term is just one needs for cosmological constant and the second one is for the mass term. Then,

$$\phi' = \frac{1}{2} \frac{\partial W}{\partial \phi} = -u\phi. \quad (4.103)$$

Substituting $\phi = Ce^{my}$ we get $m = -u$. If we use the boundary condition that at $y = 0$, $\phi = \phi_P$, we find that $C = \phi_P$. Then we have

$$\phi = \phi_P e^{-uy}. \quad (4.104)$$

From this value of the scalar field at the TeV brane at r it is determined to be

$$\phi_T = \phi_P e^{-ur}. \quad (4.105)$$

This means that the radius is no longer arbitrary but given by,

$$r = \frac{1}{u} \ln \frac{\phi_P}{\phi_T}. \quad (4.106)$$

This is the GW mechanism. The background metric will then be obtained from the equation

$$A' = \frac{\kappa^2}{6} W(\phi_0) = k - \frac{u\kappa^2}{6} \phi_P^2 e^{-2uy}, \quad (4.107)$$

given by the solution

$$A(y) = ky + \frac{\kappa^2 \phi_P^2}{12} e^{-2uy}, \quad (4.108)$$

where the first term is the usual RS warp factor, while the second term is the backreaction of the metric to the non-vanishing scalar field in the bulk. Since we want to generate the right hierarchy between the Planck and Weak scales we need to ensure that $kr \sim 30$ from which we get

$$\frac{k}{u} \ln\left(\frac{\phi_P}{\phi_T}\right) \sim 30, \quad (4.109)$$

which is the ratio that will set the hierarchy in the RS model. Since ϕ_P/ϕ_T is constant, so u is kept constant.

4.3.3 The coupled field equations and the radion mass

Once we have established the mechanism for the stabilization of radion, we realize that the radion is no longer massless. Then, the question is what will be the value of the radion mass in the GW stabilized RS model? For this we consider spin-0 fluctuations of the coupled gravity-scalar system over the background. This can be parameterized in the following way

$$\begin{aligned} ds^2 &= e^{-2A(y)-2F(x,y)} \eta_{\mu\nu} dx^\mu dx^\nu - (1 + G(x,y))^2 dy^2, \\ \phi(x,y) &= \phi_0(y) + \varphi(x,y). \end{aligned} \quad (4.110)$$

It looks like as if there would be three different fluctuations, namely F , G , φ . We will use this ansatz to linearize Einstein and scalar field equations. Then, some coupled equations for F , G , and φ will be obtained. The linearized Einstein eq. B-4 (see Appendix B for details) equations read

$$\delta R_{MN} = \kappa^2 \delta \tilde{T}_{MN}, \quad (4.111)$$

and by using this equation together with the ansatz given in eq. 4.110 where

$$\begin{aligned}
g_{\mu\nu} &= e^{-2A(y)-2F(x,y)}\eta_{\mu\nu}, \\
g^{\mu\nu} &= e^{2A(y)+2F(x,y)}\eta^{\mu\nu}, \\
g_{55} &= -(1 + G(x, y))^2, \\
g^{55} &= -(1 + G(x, y))^{-2},
\end{aligned} \tag{4.112}$$

the linearized form of the $R_{\mu\nu}$ is obtained as (see Appendix B for the detailed calculations)

$$\begin{aligned}
\delta R_{\mu\nu} &= -2\partial_\mu\partial_\nu F + \partial_\mu\partial_\nu G - \eta_{\mu\nu}\square F + e^{-2A}\eta_{\mu\nu} \\
&\times [F'' - 8A'F' - A'G' - 2FA'' + 8FA'^2 - 2GA'' + 8GA'^2]. \tag{4.113}
\end{aligned}$$

Inspecting the $\delta R_{\mu\nu}$ equation, we realize the $\partial_\mu\partial_\nu$ term must vanish since to linear order in perturbations all the terms in $\tilde{T}_{\mu\nu}$ (see eq. B-61) are proportional to $\eta_{\mu\nu}$. Then, one can immediately conclude that $G = 2F$ since

$$\delta R_{\mu\nu} = \dots - 2\partial_\mu\partial_\nu F + \partial_\mu\partial_\nu G + \dots \tag{4.114}$$

Substituting this into the equation for $\delta R_{\mu\nu}$ we get,

$$\delta R_{\mu\nu} = -\eta_{\mu\nu}\square F + \eta_{\mu\nu}e^{-2A}[F'' - 10A'F' - 6FA'' + 24A'^2F]. \tag{4.115}$$

Similarly, $\delta R_{\mu 5}$ and δR_{55} is obtained as

$$\delta R_{\mu 5} = -3\partial_\mu F' + 6A'\partial_\mu F, \tag{4.116}$$

and

$$\delta R_{55} = -4F'' - 2e^{2A}\square F + 16A'F'. \tag{4.117}$$

(see Appendix B for details). Now, we will present the linearized source terms $\delta\tilde{T}_{\mu\nu}$, $\delta\tilde{T}_{\mu 5}$, and $\delta\tilde{T}_{55}$ (see Appendix B for their derivations):

$$\begin{aligned}
\delta\tilde{T}_{\mu\nu} &= \frac{1}{3}e^{-2A}\eta_{\mu\nu}[\varphi V'(\phi_0) - 2V(\phi_0)F] \\
&\quad + \frac{1}{6}e^{-2A}\eta_{\mu\nu} \sum_i \left[\frac{\partial\lambda_i(\phi_0)}{\partial\phi} \varphi - 4\lambda_i(\phi_0)F \right] \delta(y - y_i), \\
\delta\tilde{T}_{\mu 5} &= -\frac{1}{2}\phi'_0 \partial_\mu \varphi, \\
\delta\tilde{T}_{55} &= -\frac{4}{3}V(\phi_0)F - \frac{1}{3}\varphi V'(\phi_0) - \varphi' \phi'_0 \\
&\quad - \frac{2}{3} \sum_i \left[\frac{\partial\lambda_i(\phi_0)}{\partial\phi} \varphi + 2\lambda_i(\phi_0)F \right] \delta(y - y_i).
\end{aligned} \tag{4.118}$$

Finally, we present the linearized scalar field equation for completeness:

$$\begin{aligned}
e^{2A}\square\varphi - \varphi'' + 4A'\varphi' + \frac{\partial^2 V(\phi_0)}{\partial\phi^2}\varphi &= - \sum_i \left(\frac{\partial^2\lambda_i(\phi_0)}{\partial\phi^2}\varphi + 2\frac{\partial\lambda_i(\phi_0)}{\partial\phi}F \right) \delta(y - y_i) \\
&\quad - 6\phi'_0 F' - 4\frac{\partial V}{\partial\phi}F.
\end{aligned} \tag{4.119}$$

Using the equation $\delta R_{\mu 5} = \kappa^2 \delta\tilde{T}_{\mu 5}$ we get

$$3(\partial_\mu F' - 2A'\partial_\mu F) = \kappa^2 \phi'_0 \partial_\mu \varphi. \tag{4.120}$$

This can be integrated immediately to obtain

$$\phi'_0 \varphi = \frac{3}{\kappa^2}(F' - 2A'F) + C(y), \tag{4.121}$$

where $C(y)$ is the integration constant. We set this constant as zero since we require that the fluctuations F and φ are also localized in x . Let us find the boundary conditions for F and φ on the two branes. Now, we use the equation

$\delta R_{55} = \kappa^2 \delta \tilde{T}_{55}$ which leads to,

$$2e^{2A}\square F + 4F'' - 16A'F' = \kappa^2 \left\{ 2\phi'_0\varphi' + \frac{2}{3}V'(\phi_0)\varphi + \frac{8}{3}V(\phi_0)F \right. \\ \left. + \frac{4}{3}\sum_i \left[\frac{\partial\lambda_i(\phi_0)}{\partial\phi} + 2\lambda_i(\phi_0)F \right] \delta(y - y_i) \right\}, \quad (4.122)$$

and, taking in to account the continuity of the metric but not its derivative, we get the equation

$$[F'] = \frac{2\kappa^2}{3}\lambda_i(\phi_0)F + \frac{\kappa^2}{3}\frac{\partial\lambda_i(\phi_0)}{\partial\phi}\varphi. \quad (4.123)$$

Here the delta function is proportional to the jump of the derivative of F' ,

$$F''(y=0) \sim [F'(0+\epsilon) - F'(0-\epsilon)]\delta(y), \quad (4.124)$$

with

$$F'(y=0) \sim [F'(0+\epsilon) - F'(0-\epsilon)]\varepsilon(y), \quad (4.125)$$

where $\varepsilon(y)$ is a unitstep function. In the same manner, by using linearized scalar field equation, eq. 4.119, we can take

$$[\varphi']|_i = \frac{\partial^2\lambda_i(\phi_0)}{\partial\phi^2}\varphi + 2\frac{\partial\lambda_i}{\partial\phi}F. \quad (4.126)$$

Using the jump equations, eqs. 4.98 and 4.121, with $C(y) = 0$ we get

$$[F'] = \frac{\kappa^2}{3}[\phi'_0]\varphi + 2[A']F \\ = \frac{2\kappa^2}{3}\lambda_i(\phi_0)F + \frac{\kappa^2}{3}\frac{\partial\lambda_i(\phi_0)}{\partial\phi}\varphi. \quad (4.127)$$

Thus, it provides no new constraints. Then, only the second boundary condition must be taken into account. For a convenient limit $\frac{\partial^2\lambda_i(\phi_0)}{\partial\phi^2} \gg 1$, the second boundary condition is simply $\varphi|_i = 0$. Then, in this limit the first boundary condition is just

$$(F' - 2A'F)|_i = 0. \quad (4.128)$$

Now a single equation for F is obtained as follows. Considering the combination $e^{2A}\delta R_{\mu\nu} + \eta_{\mu\nu}\delta R_{55}$ in the bulk we get

$$e^{2A}\delta R_{\mu\nu} + \eta_{\mu\nu}\delta R_{55} = 3\eta_{\mu\nu}[e^{2A}\square F + F'' - 2A'F' + 2F(A'' - 4A'^2)], \quad (4.129)$$

where $A'' - 4A'^2 = \frac{2\kappa^2}{3}V(\phi_0)$. Here, we do not take into account the δ terms since we work in the bulk. The similar combination for the source terms reads

$$e^{2A}\kappa^2\delta\tilde{T}_{\mu\nu} + \eta_{\mu\nu}\kappa^2\delta\tilde{T}_{55} = 4\kappa^2\eta_{\mu\nu}V(\phi_0)F + 2\kappa^2\eta_{\mu\nu}\phi'_0\varphi', \quad (4.130)$$

and equating them to get

$$e^{2A}\square F + F'' - 2A'F' = \frac{2\kappa^2}{3}\phi'_0\varphi'. \quad (4.131)$$

Using eq. 4.121, φ' can be obtained in terms of F as

$$\varphi = \frac{3}{\kappa^2} \frac{F' - 2A'F}{\phi'_0}. \quad (4.132)$$

If we take one more derivative with respect to y we get

$$\varphi' = \frac{3}{\kappa^2} \frac{(F'' - 2A'F' - 2A''F)\phi'_0 - (F' - 2A'F)\phi''_0}{\phi'^2_0}, \quad (4.133)$$

and by making some arrangements in this equation we obtain,

$$\frac{2\kappa^2}{3}\phi'_0\varphi' = 2(F'' - 2A'F' - 2A''F) - 2(F' - 2A'F)\frac{\phi''_0}{\phi'_0}. \quad (4.134)$$

If we substitute into the eq. 4.131 we get

$$F'' - 2A'F' - 4A''F - 2\frac{\phi''_0}{\phi'_0}F' + 4A'\frac{\phi''_0}{\phi'_0} = e^{2A}\square F, \quad (4.135)$$

to be solved in the bulk. It is important to note that each eigenmode $\square F_n = -m_n^2 F_n$ to this equation has two integration constants and one mass eigenvalue. The first constant corresponds to the overall normalization. while the remaining

one is fixed by the boundary condition on the Planck brane, and the mass is determined by the boundary condition on the TeV brane.

Now we are ready to calculate the radion mass. In the following we will show how backreaction generates a non-vanishing mass for the radion field by using eq. 4.135. Substituting $\phi_0(y) = \phi_P e^{-uy}$ into the above equation we get

$$F'' - 2A'F' - 4A''F + 2uF' - 4uA'F + m^2 e^{2A} F = 0, \quad (4.136)$$

where $A(y)$ is given in eq. 4.108. The backreaction will be treated as perturbation such that

$$A(y) = k|y| + \frac{l^2}{6} e^{-2u|y|}, \quad (4.137)$$

where $l = \kappa\phi_0/\sqrt{2}$. Then, we will look the solution in terms of the perturbative series in l . The solution is written as follows

$$F_0 = e^{2k|y|}(1 + l^2 f_0(y)) \quad ; \quad m_r^2 = l^2 \tilde{m}^2. \quad (4.138)$$

By substituting the solution into the eq. 4.136 and keeping only the leading terms in l^2 we get

$$f_0'' + 2(k+u)f_0' = -\tilde{m}^2 e^{2k|y|} - \frac{4}{3}(k-u)u e^{-2u|y|}. \quad (4.139)$$

By solving this equation we get

$$f_0'(y) = C e^{-2(k+u)|y|} - \frac{\tilde{m}^2}{2(2k+u)} e^{2k|y|} - 2 \frac{(k-u)u}{3k} e^{-2u|y|}. \quad (4.140)$$

If we make the same substitution in the boundary condition $F' - 2A'F = 0$ we get

$$l^2 f_0' + \frac{2}{3} u l^2 e^{-2u|y|} + \frac{4}{3} u l^4 f_0 e^{-2u|y|} = 0, \quad (4.141)$$

and by keeping only the leading terms in l^2 we obtain

$$f_0' + \frac{2}{3} u e^{-2u|y|} = 0. \quad (4.142)$$

at the boundary, where $|y| = r_0$. If we substitute this into the eq. 4.139, \tilde{m} is

obtained as

$$\tilde{m}^2 = \frac{4}{3} \frac{2k+u}{k} u^2 e^{-2(u+k)r_0}, \quad (4.143)$$

and the radion mass reads

$$m_{radion}^2 = \frac{4l^2(2k+u)u^2}{3k} e^{-2(u+k)r_0}. \quad (4.144)$$

4.3.4 Coupling to SM fields and the normalized radion field

In the previous section, including the backreaction, a mass scale $\mathcal{O}(TeV^2)$ is obtained for the radion mass. Then, the wavefunction can be written as

$$F_0(x, y) = e^{2k|y|} (1 + l^2 f_0(y)) R(x), \quad (4.145)$$

where $f_0(y)$ obtained using the integral in the eq. 4.140. Based on the assumption $l^2 \ll 1$, we see that the backreaction induces a small correction to the unperturbed wavefunction. So for purposes of determining the coupling of the radion to the TeV brane, we can use the unperturbed wavefunction, namely $F(x, y) = e^{2k|y|} R(x)$. Let us try to find the coefficient of $(\partial F)^2$ term in the Lagrangian $\sqrt{g}R$ with $\sqrt{g} = e^{-4A-4F}(1+2F)$, so that we are able to write the normalized wave function $R(x)$. R can be splited as $R = g^{\mu\nu} R_{\mu\nu} + g^{55} R_{55}$. Then, using the $R_{\mu\nu}$ (see Appendix B for details) read

$$\begin{aligned} R_{\mu\nu} &= -2\partial_\mu\partial_\nu F + 2\frac{\partial_\mu\partial_\nu F}{1+2F} - \eta_{\mu\nu}\Box F + \eta_{\mu\nu}\frac{e^{-2A-2F}}{(1+2F)^2} \\ &\times [A'' + F'' - 4(A' + F')^2 - \frac{2(A' + F')F'}{(1+2F)}] \\ &+ 4\frac{\partial_\mu F\partial_\nu F}{1+2F} - 2\eta_{\mu\nu}\frac{(\partial F)^2}{1+2F} - 2\partial_\mu F\partial_\nu F + 2\eta_{\mu\nu}(\partial F)^2. \end{aligned} \quad (4.146)$$

and multiplying by $\sqrt{g}g^{\mu\nu}$ from left, for the first term in R , including $(\partial F)^2$, we get

$$\begin{aligned}\sqrt{g}g^{\mu\nu}R_{\mu\nu} &\cong e^{-2A}(1-2F)(-4(1+2F)F\Box F - 4(1+2F)\Box F + 6(1+2F)(\partial F)^2 \\ &\quad - 4(\partial F)^2 \dots),\end{aligned}\tag{4.147}$$

where $F\Box F = F\partial_\mu\partial^\mu F = \partial_\mu(F\partial^\mu F) - (\partial F)^2$. Substituting this quality into the above equation we obtain,

$$\begin{aligned}g^{\mu\nu}R_{\mu\nu} &\cong e^{-2A}(1-2F)(-4[\partial_\mu(F\partial^\mu F) - (\partial F)^2](1+2F) \\ &\quad - 4(\partial_\mu(F\partial^\mu F) - (\partial F)^2)F - 8[\partial_\mu(F\partial^\mu F) - (\partial F)^2] \\ &\quad - 2(\partial F)^2 + \dots).\end{aligned}\tag{4.148}$$

Similarly, using the equation for R_{55} (see Appendix B for details) below

$$\begin{aligned}R_{55} &= -4(A'' + F'') + 4e^{2A+2F}(1+2F)(\partial F)^2 \\ &\quad - 2e^{2A+2F}\Box F(1+2F) + \frac{8(A' + F')F'}{1+2F} + 4(A' + F')^2,\end{aligned}\tag{4.149}$$

the second term in R is obtained as

$$\begin{aligned}\sqrt{g}g^{55}R_{55} &= \frac{-e^{-4A-4F}(1+2F)}{(1+2F)^2}R_{55} \\ &= e^{-4A-4F}(1+2F)\left(\frac{4(A'' + F'')}{(1+2F)^2} - \frac{4e^{2A+2F}(\partial F)^2}{1+2F} + \frac{2e^{2A+2F}\Box F}{1+2F} \right. \\ &\quad \left. - \frac{8(A' + F')F'}{(1+2F)^3} + \frac{4(A' + F')^2}{(1+2F)^2}\right) \\ &\cong e^{-4A}(1-2F)\left(4(A'' + F'')(1-4F) - 4e^{2A}(1-4F^2)(\partial F)^2 \right. \\ &\quad \left. + 2e^{2A}(1-4F^2)(\partial_\mu(F\partial^\mu F) - (\partial F)^2)F - 8(A' + F')F'(1-6F) \right. \\ &\quad \left. - 4(A' + F')^2(1-4F)\right).\end{aligned}\tag{4.150}$$

So, one can simply find the coefficient of $(\partial F)^2$ in $\sqrt{g}R$ in linear order as $e^{-2A}6$. Then, a straightforward calculation gives

$$-M_*^3 \int dy \sqrt{g} R \supset 6M_*^3 (\partial R)^2 \int e^{-2A} e^{4k|y|} = \frac{6M_*^3}{k} (e^{2kr_0} - 1) (\partial R)^2. \quad (4.151)$$

Therefore, the normalized radion $r(x)$ is $R(x) = r(x)e^{-kr_0}/\sqrt{6}M_{Pl}$, which is obtained by using $M_*^3/k = M_{Pl}^2/2$.

CHAPTER 5

LEPTON FLAVOR VIOLATING RADION DECAYS IN THE RANDALL-SUNDRUM SCENARIO

The hierarchy problem between weak and Planck scales could be explained by introducing the extra dimensions. One of the possibility is to pull down the Planck scale to TeV range by considering the compactified extra dimensions of large size [20, 21]. The assumption that the extra dimensions are at the order of submillimeter distance, for two extra dimensions, the hierarchy problem in the fundamental scales could be solved and the true scale of quantum gravity would be no more the Planck scale but it is of the order of EW scale. This is the case that the gravity spreads over all the volume including the extra dimensions, however, the matter fields are restricted in four dimensions, so called 4D brane. Another possibility, which is based on the non-factorizable geometry, is introduced by Randall and Sundrum [30, 31] and, in this scenario, the extra dimension is compactified to S^1/Z_2 orbifold with two 4D brane boundaries. Here, the gravity is localized in one of the boundary, so called the Planck brane, which is away from another boundary, the TeV brane where we live. The size of extra dimension is related to the vacuum expectation of a scalar field and its fluctuation over the expectation value is called the radion field (see section 2 for details). The radion in the RS1 model has been studied in several works in the literature [50]-[58] (see [47] for extensive discussion).

In the present work, [63] we study the possible LFV decays of the radion field

r . The LFV interactions exist at least in one loop level in the extended SM, so called ν SM, which is constructed by taking neutrinos massive and by permitting the lepton mixing mechanism [41]. Their negligibly small Brs stimulate one to go beyond and they are worthwhile to examine since they open a window to test new models and to ensure considerable information about the restrictions of the free parameters, with the help of the possible accurate measurements. The LFV interactions are carried by the FCNCs and in the SM with extended Higgs sector (the multi Higgs doublet model) they can exist at tree level. Among multi Higgs doublet models, the 2HDM is a candidate for the lepton flavor violation. In this model, the lepton flavor violation is driven by the new scalar Higgs bosons S , scalar h^0 and pseudo scalar A^0 , and it is controlled by the Yukawa couplings appearing in lepton-lepton- S vertices.

Here, we predict the BRs of the LFV r decays in the 2HDM, in the framework of the RS1 scenario. We observe that the BRs of the processes we study are at most of the order of 10^{-8} , for the small values of radion mass m_r and their sensitivities to m_r decrease with the increasing values of m_r . Among the LFV decays we study, the $r \rightarrow \tau^\pm \mu^\pm$ decay would be the most suitable one to measure its BR.

5.1 The LFV RS1 model radion decay in the 2HDM

The RS1 model is an interesting candidate in order to explain the well known hierarchy problem. It is formulated as two 4D surfaces (branes) in 5D world in which the extra dimension is compactified into S^1/Z_2 orbifold. In this model, the SM fields are assumed to live on one of the brane, so called the TeV brane. On the other hand, the gravity peaks near the other brane, so called the Planck brane and extends into the bulk with varying strength. Here, 5D cosmological constant is non vanishing and both branes have equal and opposite tensions so that the low energy effective theory has flat 4D spacetime. The metric of such

5D world reads

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \quad (5.1)$$

where $A(y) = k|y|$, k is the bulk curvature constant, y is the extra dimension parametrized as $y = R\theta$. The exponential factor e^{-kL} with $L = R\pi$, is the warp factor which causes that all the mass terms are rescaled in the TeV brane. With a rough estimate $L \sim 30/k$, all mass terms are brought down to the TeV scale. The size L of extra dimension is related to the vacuum expectation of the field $L(x)$ and its fluctuation over the expectation value is called the radion field r . In order to avoid the violation of equivalence principle, $L(x)$ should acquire a mass and, to stabilize r , a mechanism was proposed by Goldberger and Wise [50], by introducing a potential for $L(x)$. Finally the metric in 5D is defined as [51].

$$ds^2 = e^{-2A(y)-2F(x)} \eta_{\mu\nu} dx^\mu dx^\nu - (1 + 2F(x)) dy^2, \quad (5.2)$$

where the radial fluctuations are carried by the scalar field $F(x)$,

$$F(x) = \frac{1}{\sqrt{6} M_{Pl} e^{-kL}} r(x). \quad (5.3)$$

Here the field $r(x)$ is the normalized radion field (see [48]). At the orbifold point $\theta = \pi$ (TeV brane) the induced metric reads,

$$g_{\mu\nu}^{ind} = e^{-2A(L)-2\frac{\gamma}{v}r(x)} \eta_{\mu\nu}. \quad (5.4)$$

Here the parameter γ reads $\gamma = \frac{v}{\sqrt{6}\Lambda}$ with $\Lambda = M_{Pl} e^{-kL}$ and v is the vacuum expectation value of the SM Higgs boson. The radion is the additional degree of freedom of the 4D effective theory and we study the possible LFV decays of this field.

The FCNCs at tree level can exist in the 2HDM and they induce the FV interactions with large BRs. The FV r decays, $r \rightarrow l_1^- l_2^+$, can exist at least in one loop level in the framework of the 2HDM. The part of action which carries

the interaction, responsible for the LFV processes reads

$$\mathcal{S}_Y = \int d^4x \sqrt{-g^{ind}} \left(\eta_{ij}^E \bar{l}_{iL} \phi_1 E_{jR} + \xi_{ij}^E \bar{l}_{iL} \phi_2 E_{jR} + h.c. \right), \quad (5.5)$$

where L and R denote chiral projections $L(R) = 1/2(1 \mp \gamma_5)$, ϕ_i for $i = 1, 2$, are two scalar doublets, l_{iL} (E_{jR}) are lepton doublets (singlets), $\xi_{ij}^{E\ 1}$ and η_{ij}^E , with family indices i, j , are the Yukawa couplings and ξ_{ij}^E induce the FV interactions in the leptonic sector. Here g^{ind} is the determinant of the induced metric on the TeV brane where the 2HDM particles live. Here, we assume that the Higgs doublet ϕ_1 has a non-zero vacuum expectation value to ensure the ordinary masses of the gauge fields and the fermions, however, the second doublet has no vacuum expectation value, namely, we choose the doublets ϕ_1 and ϕ_2 and their vacuum expectation values as

$$\phi_1 = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ v + H^0 \end{pmatrix} + \begin{pmatrix} \sqrt{2}\chi^+ \\ i\chi^0 \end{pmatrix} \right]; \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}H^+ \\ H_1 + iH_2 \end{pmatrix}, \quad (5.6)$$

and

$$\langle \phi_1 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}; \langle \phi_2 \rangle = 0. \quad (5.7)$$

This choice ensures that the mixing between neutral scalar Higgs bosons is switched off and it would be possible to separate the particle spectrum so that the SM particles are collected in the first doublet and the new particles in the second one². The action in eq. (5.5) is responsible for the tree level $S - l_1 - l_2$ (l_1 and l_2 are different flavors of charged leptons, S denotes the neutral new Higgs boson, $S = h^0, A^0$) interaction (see Fig. 5.3-d, e) and the four point $r - S - l_1 - l_2$ interaction (see Fig. 5.3-c) where r is the radion field. The latter interaction is coming from the determinant factor $\sqrt{-g^{ind}} = e^{-4A(L) - 4\frac{\gamma}{v}r(x)}$. Notice that the term $e^{-4A(L)}$ in $\sqrt{-g^{ind}}$ is embedded into the redefinitions of the fields on the

¹In the following, we replace ξ^E with ξ_N^E where "N" denotes the word "neutral".

²Here H_1 (H_2) is the well known mass eigenstate h^0 (A^0).

TeV brane, namely, they are warped as $S \rightarrow e^{A(L)} S_{\text{warp}}$, $l \rightarrow e^{3A(L)/2} l_{\text{warp}}$ and in the following we use warped fields without the *warp* index.

On the other hand, the part of new scalar action

$$\mathcal{S}_2 = \int d^4x \sqrt{-g^{\text{ind}}} \left(g^{\text{ind} \mu \nu} (D_\mu \phi_2)^\dagger D_\nu \phi_2 - m_S^2 \phi_2^\dagger \phi_2 \right) \quad (5.8)$$

leads to

$$\begin{aligned} \mathcal{S}'_2 = & \frac{1}{2} \int d^4x \left\{ e^{-2\frac{\gamma}{v} r} \eta^{\mu \nu} \left(\partial_\mu h^0 \partial_\nu h^0 + \partial_\mu A^0 \partial_\nu A^0 \right) \right. \\ & \left. - e^{-4\frac{\gamma}{v} r} (m_{h^0}^2 h^0 h^0 + m_{A^0}^2 A^0 A^0) \right\}, \end{aligned} \quad (5.9)$$

which carries the $S - S - r$ interaction³ (see Fig. 5.3 1-b).

Finally, the interaction of leptons with the radion field is carried by the action (see [52])

$$\mathcal{S}_3 = \int d^4x \sqrt{-g^{\text{ind}}} \left(g^{\text{ind} \mu \nu} \bar{l} \gamma_\mu i D_\nu l - m_l \bar{l} l \right), \quad (5.11)$$

where

$$D_\mu l = \partial_\mu l + \frac{1}{2} w_\mu^{ab} \Sigma_{ab} l, \quad (5.12)$$

with $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$. Here w_μ^{ab} is the spin connection and, by using the vierbein

³In general, there is no symmetry which forbids the curvature-scalar interaction,

$$\mathcal{S}_\xi = \int d^4x \sqrt{-g^{\text{ind}}} \xi \mathcal{R} H^\dagger H, \quad (5.10)$$

where ξ is a restricted positive parameter and H is the Higgs scalar field [47, 48, 59]. This interaction results in the radion-(SM or new) Higgs mixing which can bring a sizeable contribution to the physical quantities studied. Here, we assume that there is no mixing between first and second doublet and only the first Higgs doublet has vacuum expectation value. Therefore, we choose that there exists a mixing between the radion and the SM Higgs field, but not between the radion and the new Higgs fields. This is the case that the lepton flavor violation is not affected by the mixing since the SM Higgs field is not responsible for the FCNC current at tree level.

fields e_μ^a , it can be calculated (linear in r) as

$$w_\mu^{ab} = -\frac{\gamma}{v} \partial_\nu r (e^{\nu b} e_\mu^a - e^{\nu a} e_\mu^b), \quad (5.13)$$

(see Appendix C for details). Notice that the vierbein fields are the square root of the metric and they satisfy the relation

$$e_a^\mu e^{a\nu} = g^{ind\ \mu\nu}. \quad (5.14)$$

Using eqs. (5.11)-(5.14), one gets the part of the action which describes the tree level $l-l-r$ interaction (see Fig. 5.3-a) as

$$\mathcal{S}'_3 = \int d^4x \left\{ -3\frac{\gamma}{v} r \bar{l} i \not{\partial} l - 3\frac{\gamma}{2v} \bar{l} i \not{\partial} r l + 4\frac{\gamma}{v} m_l r \bar{l} l \right\}. \quad (5.15)$$

Now, we are ready to calculate the matrix element for the LFV radion decay. The decay of the radion to leptons with different flavors exits at least in one loop order, with the help of internal new Higgs bosons $S = h^0, A^0$. The possible vertex and self energy diagrams are presented in Fig. 5.2. After addition of all these diagrams, the divergences which occur in the loop integrals are eliminated and the matrix element square for this decay is obtained as

$$|M|^2 = 2 \left(m_r^2 - (m_{l_1^-} + m_{l_2^+})^2 \right) |A|^2, \quad (5.16)$$

where

$$A = f_{h^0}^{self} + f_{A^0}^{self} + f_{h^0}^{vert} + f_{A^0}^{vert} + f_{h^0 h^0}^{vert} + f_{A^0 A^0}^{vert}, \quad (5.17)$$

and their explicit expressions are given by

$$\begin{aligned}
f_{h^0}^{self} &= \frac{\gamma}{128 v \pi^2 (w'_h - w_h)} \int_0^1 dx m_{h^0} \left\{ \left(\eta_i^V (x-1) w_h - \eta_i^+ z_{ih} \right) \right. \\
&\times \left(3 w'_h - 5 w_h \right) \ln \frac{L_{1,h^0}^{self} m_{h^0}^2}{\mu^2} \\
&+ \left. \left(\eta_i^V (x-1) w'_h - \eta_i^+ z_{ih} \right) \left(5 w'_h - 3 w_h \right) \ln \frac{L_{2,h^0}^{self} m_{h^0}^2}{\mu^2} \right\}, \\
f_{A^0}^{self} &= \frac{\gamma}{128 v \pi^2 (w'_A - w_A)} \int_0^1 dx, m_{A^0} \left\{ \left(\eta_i^V (x-1) w_A + \eta_i^+ z_{iA} \right) \right. \\
&\times \left(3 w'_A - 5 w_A \right) \ln \frac{L_{1,A^0}^{self} m_{A^0}^2}{\mu^2} \\
&+ \left. \left(\eta_i^V (x-1) w'_A + \eta_i^+ z_{iA} \right) \left(5 w'_A - 3 w_A \right) \ln \frac{L_{2,A^0}^{self} m_{A^0}^2}{\mu^2} \right\}, \\
f_{h^0}^{vert} &= \frac{\gamma}{128 v \pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{m_{h^0}}{L_{h^0}^{ver}} \left[\eta_i^V \left(3 z_{rh}^2 \left(y(1-y) w'_h + x^2 (4y-1) w_h \right) \right. \right. \right. \\
&+ x \left((1-3y) w_h + y(4y-3) w'_h \right) + 5 z_i^2 \left((2y-1) w'_h + (2x-1) w_h \right) \\
&- 3(x+y-1) \left(x(4x-3) w_h^3 + y(4y-3) w_h'^3 \right) \\
&- 3 w_h w'_h (x+y-1) \left((1-y+x(4y-2)) w_h + (1-2y+x(4y-1)) w'_h \right) \\
&+ \left. \left. \left. 3(x+y-1) \left((2x-1) w_h + (2y-1) w'_h \right) \right) \right) \right. \\
&+ \eta_i^+ z_{ih} \left((x+y-1) \left(-4 + 2 w'_h w_h + w_h'^2 (8y-3) + w_h^2 (8x-3) \right) \right. \\
&- \left. \left. \left. \left(8 z_{ih}^2 + z_{rh}^2 ((8y-3)x - 3y) \right) \right) \right) \right] \\
&- \left. m_{h^0} \ln \frac{L_{h^0}^{ver} m_{h^0}^2}{\mu^2} \left(9 \eta_i^V \left(w'_h (2y-1) + w_h (2x-1) \right) - 8 \eta_i^+ z_{ih} \right) \right\},
\end{aligned}$$

$$\begin{aligned}
f_{A^0}^{vert} &= \frac{\gamma}{128 v \pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{m_{A^0}}{L_{A^0}^{ver}} \left[\eta_i^V \left(3 z_{rA}^2 \left(y(1-y) w'_A + x^2 (4y-1) w_A \right. \right. \right. \right. \\
&+ x((1-3y) w_A + y(4y-3) w'_A) \Big) + 5 z_i^2 \left((2y-1) w'_A + (2x-1) w_A \right) \\
&- 3(x+y-1) \left(x(4x-3) w_A^3 + y(4y-3) w_A'^3 \right) \\
&- 3 w_A w'_A (x+y-1) \left((1-y+x(4y-2)) w_A + (1-2y+x(4y-1)) w'_A \right) \\
&+ 3(x+y-1) \left((2x-1) w_A + (2y-1) w'_A \right) \Big) \\
&+ \eta_i^+ z_{iA} \left((x+y-1) \left(-4 + 2 w'_A w_A + w_A'^2 (8y-3) + w_A^2 (8x-3) \right) \right. \\
&- \left. \left(8 z_{iA}^2 + z_{rA}^2 ((8y-3)x - 3y) \right) \right) \Big] \\
&- m_{A^0} \ln \frac{L_{A^0}^{ver} m_{A^0}^2}{\mu^2} \left(9 \eta_i^V \left(w'_A (2y-1) + w_A (2x-1) \right) + 8 \eta_i^+ z_{iA} \right) \Big\}, \\
f_{h^0 h^0}^{vert} &= \frac{\gamma}{64 v \pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{m_{h^0}}{L_{h^0 h^0}^{ver}} \left[\eta_i^V \left(z_{rh}^2 \left(y-1+x(1-4y) \right) (x w_h + y w'_h) \right. \right. \right. \\
&+ y(x+y-1) w'_h \left((4x-1) w_h^2 + (4y-1) w_h'^2 \right) \\
&+ w_h^3 x(x+y-1)(4x-1) + (x+y-1) \left(2y w'_h + x w_h (2 + w_h'^2 (4y-1)) \right) \Big) \\
&+ \eta_i^+ \left((x+y-1) z_{ih} \left((4y-1) w_h'^2 + (4x-1) w_h^2 + 2 \right) \right. \\
&- \left. z_{ih} z_{rh}^2 \left((4y-1)x - y + 1 \right) \right) \Big] \\
&- m_{h^0} \ln \frac{L_{h^0 h^0}^{ver} m_{h^0}^2}{\mu^2} \left(\eta_i^V \left(w'_h (1-6y) + w_h (1-6x) \right) - 4 \eta_i^+ z_{ih} \right) \Big\},
\end{aligned}$$

$$\begin{aligned}
f_{A^0 A^0}^{vert} = & \frac{\gamma}{64 v \pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{m_{A^0}}{L_{A^0 A^0}^{ver}} \left[\eta_i^V \left(z_{rA}^2 (y-1+x(1-4y)) (x w_A + y w'_A) \right. \right. \right. \\
& + y (x+y-1) w'_A \left((4x-1) w_A^2 + (4y-1) w_A'^2 \right) \\
& + w_A^3 x (x+y-1) (4x-1) + (x+y-1) \left(2y w'_A + x w_A (2 + w_A'^2 (4y-1)) \right) \Big) \\
& + \eta_i^+ \left((x+y-1) z_{iA} \left((4y-1) w_A'^2 + (4x-1) w_A^2 + 2 \right) \right. \\
& \left. \left. - z_{iA} z_{rA}^2 \left((4y-1)x - y + 1 \right) \right) \right] \\
& - m_{A^0} \ln \frac{L_{A^0 A^0}^{ver} m_{A^0}^2}{\mu^2} \left(\eta_i^V \left(w'_A (1-6y) + w_A (1-6x) \right) + 4 \eta_i^+ z_{iA} \right) \Big\}, \quad (5.18)
\end{aligned}$$

where

$$\begin{aligned}
L_{1,h^0(A^0)}^{self} &= 1 + x^2 w_{h(A)}^2 + x (z_{ih(iA)}^2 - w_{h(A)}^2 - 1), \\
L_{2,h^0(A^0)}^{self} &= 1 + x^2 w_{h(A)}'^2 + x (z_{ih(iA)}^2 - w_{h(A)}'^2 - 1), \\
L_{h^0(A^0)}^{ver} &= x^2 w_{h(A)}^2 + (y-1) (w_{h(A)}'^2 y - 1) + x (y w_{h(A)}'^2 + (y-1) w_{h(A)}^2 - y z_{rh(rA)}^2 - 1), \\
L_{h^0 h^0(A^0 A^0)}^{ver} &= x^2 w_{h(A)}^2 + (1 + w_{h(A)}'^2 (y-1)) y + x (1 + w_{h(A)}^2 (y-1) + w_{h(A)}'^2 y - z_{rh(rA)}^2 y), \quad (5.19)
\end{aligned}$$

with the parameters $w_{h(A)} = \frac{m_{l_1^-}}{m_{h^0(A^0)}}$, $w'_{h(A)} = \frac{m_{l_2^+}}{m_{h^0(A^0)}}$, $z_{rh(rA)} = \frac{m_r}{m_{h^0(A^0)}}$, $z_{ih(iA)} = \frac{m_i}{m_{h^0(A^0)}}$ and

$$\begin{aligned}
\eta_i^V &= \xi_{N,l_i}^E \xi_{N,l_2}^{E*} + \xi_{N,l_1}^{E*} \xi_{N,l_2 i}^E, \\
\eta_i^+ &= \xi_{N,l_1}^{E*} \xi_{N,l_2}^{E*} + \xi_{N,l_1 i}^E \xi_{N,l_2 i}^E. \quad (5.20)
\end{aligned}$$

In eq. (5.20), the flavor changing couplings $\xi_{N,l_j i}^E$ represent the effective interaction between the internal lepton i , ($i = e, \mu, \tau$) and the outgoing $j = 1$ ($j = 2$) lepton (anti lepton). Here, we choose the couplings $\xi_{N,l_j i}^E$ real.

Finally, the BR for $r \rightarrow l_1^- l_2^+$ can be obtained by using the matrix element

square as

$$BR(r \rightarrow l_1^- l_2^+) = \frac{1}{16 \pi m_r} \frac{|M|^2}{\Gamma_r}, \quad (5.21)$$

where Γ_r is the total decay width of radion r . In our numerical analysis, we consider the BR due to the production of sum of charged states, namely

$$BR(r \rightarrow l_1^\pm l_2^\pm) = \frac{\Gamma(r \rightarrow (\bar{l}_1 l_2 + \bar{l}_2 l_1))}{\Gamma_r}. \quad (5.22)$$

5.2 Numerical Analysis and Discussion

In four dimensions, the higher dimensional gravity is observed as it has new states with spin 2,1 and 0, so called, the graviton, the gravivector, the graviscalar. These states interact with the particles in the underlying theory. In the RS1 model with one extra dimension, the spin 0 gravity particle radion r interacts with the particles of the theory (2HDM in our case) on the TeV brane and this interaction occurs over the trace of the energy-momentum tensor T_μ^μ with the strength $1/\Lambda_r$,

$$\mathcal{L}_{int} = \frac{r}{\Lambda_r} T_\mu^\mu, \quad (5.23)$$

where Λ_r is at the order of TeV. The radion interacts with gluon (g) pair or photon (γ) pair in one loop order from the trace anomaly. For the radion mass $m_r \leq 150 \text{ GeV}$, the decay width is dominated by $r \rightarrow gg$. For the masses which are beyond the WW and ZZ thresholds, the main decay mode is $r \rightarrow WW$. In the present work, we study the possible LFV decays of the RS1 radion in the 2HDM and estimate the BRs of these decays for different values of radion masses. We take the total decay width Γ_r of the radion by considering the dominant decays $r \rightarrow gg (\gamma\gamma, ff, W^+W^-, ZZ, SS)$ where S are the neutral Higgs particles (see [55] for the explicit expressions of these decay widths). Here, we include the possible processes in the Γ_r according to the mass of the radion.

The flavor violating r decays $r \rightarrow l_1^- l_2^+$ can exist at least in one loop level, in the framework of the 2HDM and the flavor violation is carried by the Yukawa

couplings $\bar{\xi}_{N,ij}^E$ ⁴. In the version of 2HDM where the FCNC are permitted, these couplings are free parameters which should be restricted by using the present and forthcoming experiments. At first, we assume that these couplings are symmetric with respect to the flavor indices i and j . Furthermore, we take that the couplings which contain at least one τ index are dominant and we choose a broad range for these couplings, by respecting the upper limit prediction of $\bar{\xi}_{N,\tau\mu}^E$ (see [60] and references therein) which is obtained by using the experimental uncertainty, 10^{-9} , in the measurement of the muon anomalous magnetic moment and by assuming that the new physics effects can not exceed this uncertainty. For the coupling $\bar{\xi}_{N,\tau e}^E$, the restriction is estimated by using this upper limit and the experimental upper bound of BR of $\mu \rightarrow e\gamma$ decay, $\text{BR} \leq 1.2 \times 10^{-11}$ [61]. Finally, this coupling is taken in the range $10^{-3} - 10^{-1} \text{ GeV}$ (see [62]). For the Yukawa coupling $\bar{\xi}_{N,\tau\tau}^E$, we have no explicit restriction region and we use the numerical values which are greater than $\bar{\xi}_{N,\tau\mu}^E$. Throughout our calculations we use the input values given in Table (5.1).

Table 5.1: The values of the input parameters used in the numerical calculations.

Parameter	Value
m_μ	0.106 (GeV)
m_τ	1.78 (GeV)
m_{h^0}	100 (GeV)
m_{A^0}	200 (GeV)
G_F	$1.1663710^{-5}(\text{GeV}^{-2})$

In Fig.5.3 we present m_r dependence of the BR ($r \rightarrow \tau^\pm \mu^\pm$). The solid-dashed lines represent the BR ($r \rightarrow \tau^\pm \mu^\pm$) for $\bar{\xi}_{N,\tau\tau}^E = 100 \text{ GeV}$, $\bar{\xi}_{N,\tau\mu}^E = 10 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 10 \text{ GeV}$, $\bar{\xi}_{N,\tau\mu}^E = 1 \text{ GeV}$. It is observed that the BR ($r \rightarrow \tau^\pm \mu^\pm$) is of the order of the magnitude of 10^{-8} for the large values of the couplings and the radion mass values $\sim 200 \text{ GeV}$. For the heavy masses of the radion the BR is stabilized to the values of the order of 10^{-9} .

⁴The dimensionfull Yukawa couplings $\bar{\xi}_{N,ij}^E$ are defined as $\xi_{N,ij}^E = \sqrt{\frac{4G_F}{\sqrt{2}}} \bar{\xi}_{N,ij}^E$.

Fig.5.4 is devoted to m_r dependence of the BR ($r \rightarrow \tau^\pm e^\pm$) and BR ($r \rightarrow \mu^\pm e^\pm$). The solid-dashed lines represent the BR ($r \rightarrow \tau^\pm e^\pm$) for $\bar{\xi}_{N,\tau\tau}^E = 100 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$ - $\bar{\xi}_{N,\tau\tau}^E = 10 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$. The small dashed line represents the BR ($r \rightarrow \mu^\pm e^\pm$) for $\bar{\xi}_{N,\tau\mu}^E = 1 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$. This figure shows that the BR ($r \rightarrow \tau^\pm \mu^\pm$) is of the order of the magnitude of 10^{-12} for the large values of the couplings and the radion mass values $\sim 200 \text{ GeV}$. For the heavy masses of the radion, this BR reaches to the values less than 10^{-14} . The BR ($r \rightarrow \mu^\pm e^\pm$) is of the order of 10^{-15} for $m_r \sim 200 \text{ GeV}$ and for the intermediate values of Yukawa couplings. These BRs, especially BR ($r \rightarrow \mu^\pm e^\pm$), are negligibly small.

Now, we present the Yukawa coupling dependencies of the BRs of the decays under consideration, for different radion masses .

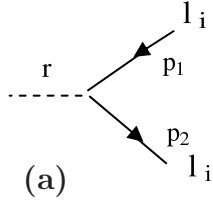
Fig.5.5 represents the $\bar{\xi}_{N,\tau\tau}^E$ dependence of the BR ($r \rightarrow \tau^\pm \mu^\pm$) for $\bar{\xi}_{N,\tau\mu}^E = 10 \text{ GeV}$. The solid-dashed-small dashed lines represent the BR for the radion masses $m_r = 200 - 500 - 1000 \text{ GeV}$. This figure shows that the BR is sensitive to the radion mass and, obviously, it is enhanced two orders of magnitude in the range $10 \text{ GeV} \leq \bar{\xi}_{N,\tau\tau}^E \leq 100 \text{ GeV}$.

In Fig.5.6, we present the $\bar{\xi}_{N,\tau\tau}^E$ dependence of the BR ($r \rightarrow \tau^\pm e^\pm$) for $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$. The solid-dashed-small dashed lines represent the BR for the radion masses $m_r = 200 - 500 - 1000 \text{ GeV}$. Similar to the $r \rightarrow \tau^\pm \mu^\pm$ decay, the BR is strongly sensitive to the radion mass.

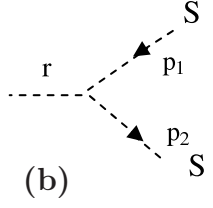
As a summary, the LFV decays of the radion in the RS1 model strongly depend on the radion mass and the Yukawa couplings. The BR for $r \rightarrow \tau^\pm \mu^\pm$ decay is of the order of 10^{-8} for the small values of radion mass m_r and it decreases with the increasing values of m_r . On the other hand, the BRs for $r \rightarrow \tau^\pm e^\pm$ ($r \rightarrow \mu^\pm e^\pm$) decays are of the order of 10^{-12} (10^{-15}) for the small values of m_r . These results show that, among these processes, the LFV $r \rightarrow \tau^\pm \mu^\pm$ decay would be the most appropriate one to measure its BR. With the possible production of the radion (the most probable production is due to the gluon fusion, $gg \rightarrow r$ [55]), hopefully, the future experimental results of this decay would be useful in order to test the possible signals coming from the extra dimensions and new physics which results in flavor violation.

5.3 The vertices appearing in the present work

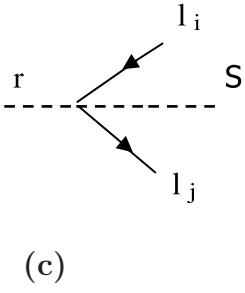
In this section we present the vertices appearing in our calculations. Here S denotes the new neutral Higgs bosons h^0 and A^0 .



$$\frac{-i\gamma}{v} \left[\frac{3}{2} (\not{p}_1 + \not{p}_2) - 4 m_{l_i} \right]$$

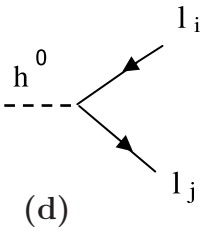


$$\frac{-2i\gamma}{v} (p_1 \cdot p_2 - m_S^2)$$

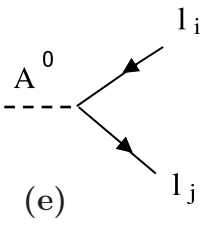


$$(S = h^0) \quad \frac{4i\gamma}{2\sqrt{2}v} [(\xi_{ij}^E + \xi_{ji}^{E*}) + (\xi_{ij}^E - \xi_{ji}^{E*})\gamma_5]$$

$$(S = A^0) \quad \frac{-4\gamma}{2\sqrt{2}v} [(\xi_{ij}^E - \xi_{ji}^{E*}) + (\xi_{ij}^E + \xi_{ji}^{E*})\gamma_5]$$

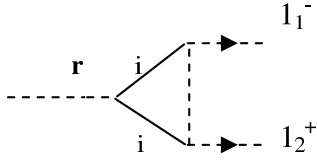


$$\frac{-i}{2\sqrt{2}} [(\xi_{ij}^E + \xi_{ji}^{E*}) + (\xi_{ij}^E - \xi_{ji}^{E*})\gamma_5]$$

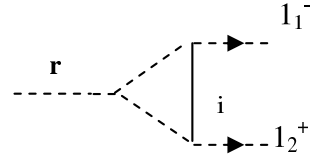


$$\frac{1}{2\sqrt{2}} [(\xi_{ij}^E - \xi_{ji}^{E*}) + (\xi_{ij}^E + \xi_{ji}^{E*})\gamma_5]$$

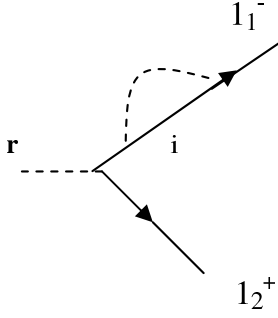
Figure 5.1: The vertices used in the present work.



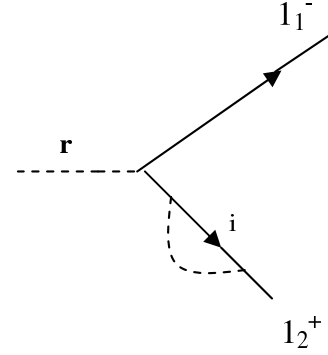
(a)



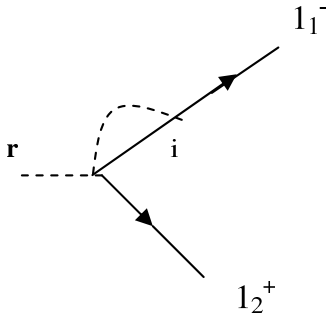
(b)



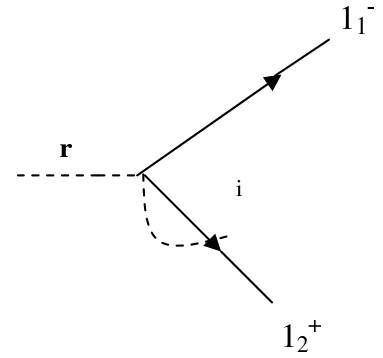
(c)



(d)



(e)



(f)

Figure 5.2: One loop diagrams contribute to $r \rightarrow l_1^- l_2^+$ decay due to the neutral Higgs bosons h_0 and A_0 in the 2HDM. i represents the internal lepton, l_1^- (l_2^+) outgoing lepton (anti lepton), internal dashed line the h_0 and A_0 fields.

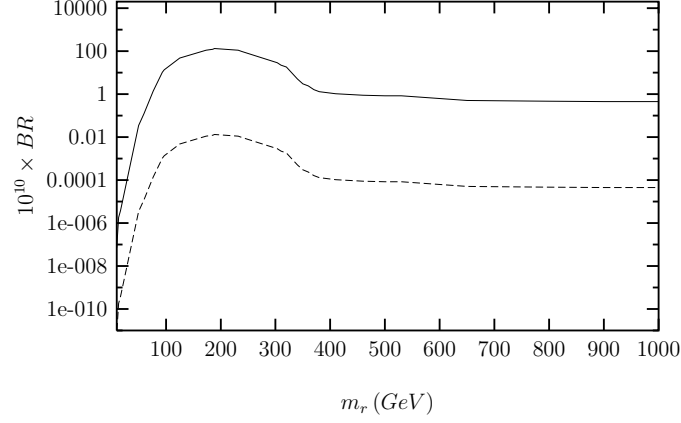


Figure 5.3: m_r dependence of the BR ($r \rightarrow \tau^\pm \mu^\pm$). The solid-dashed lines represent the BR($r \rightarrow \tau^\pm \mu^\pm$) for $\bar{\xi}_{N,\tau\tau}^E = 100 \text{ GeV}$, $\bar{\xi}_{N,\tau\mu}^E = 10 \text{ GeV}$ - $\bar{\xi}_{N,\tau\tau}^E = 10 \text{ GeV}$, $\bar{\xi}_{N,\tau\mu}^E = 1 \text{ GeV}$.

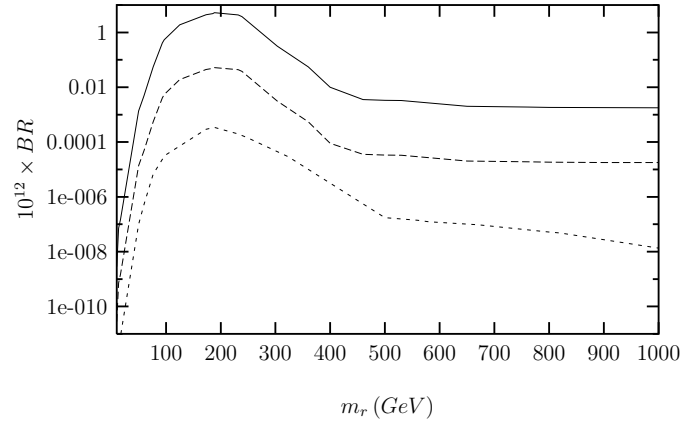


Figure 5.4: m_r dependence of the BR ($r \rightarrow l_1^\pm l_2^\pm$). The solid-dashed lines represent the BR($r \rightarrow \tau^\pm e^\pm$) for $\bar{\xi}_{N,\tau\tau}^E = 100 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$ - $\bar{\xi}_{N,\tau\tau}^E = 10 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$. The small dashed line represents the BR ($r \rightarrow \mu^\pm e^\pm$) for $\bar{\xi}_{N,\tau\mu}^E = 1 \text{ GeV}$, $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$.

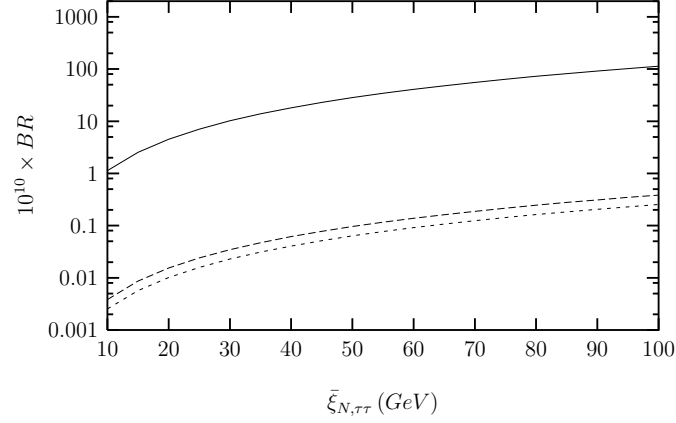


Figure 5.5: $\bar{\xi}_{N,\tau\tau}^E$ dependence of the $\text{BR}(r \rightarrow \tau^\pm \mu^\pm)$ for $\bar{\xi}_{N,\tau\mu}^E = 10 \text{ GeV}$. The solid-dashed-small dashed lines represent the BR for the radion masses $m_r = 200 - 500 - 1000 \text{ GeV}$.

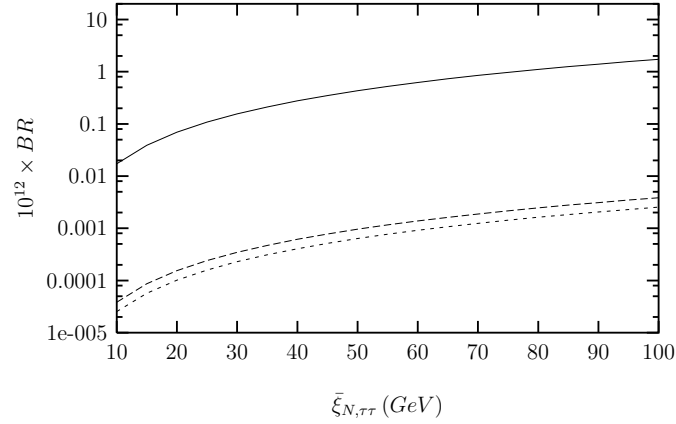


Figure 5.6: $\bar{\xi}_{N,\tau\tau}^E$ dependence of the $\text{BR}(r \rightarrow \tau^\pm e^\pm)$ for $\bar{\xi}_{N,\tau e}^E = 0.1 \text{ GeV}$. The solid-dashed-small dashed lines represent the BR for the radion masses $m_r = 200 - 500 - 1000 \text{ GeV}$.

CHAPTER 6

CONCLUSION

Over the last thirty years or so, the SM has been subjected to many diverse experiments and most of these experiments have been found to be consistent with the predictions of the SM except the recently performed ones on the FCNCs including LFV interactions. In the framework of the SM, LFV interactions exist in the ν SM by taking the neutrinos massive and permitting the lepton mixing mechanism [41, 42] in order to accommodate the present data on neutrino masses and mixings. However, the neutrino masses are so small that, the predicted BRs of the LFV interactions are too small to explain the experimental data obtained. Therefore, it is clear that there must be a theory which goes beyond the SM, but which reproduces the results of the SM where the SM has been shown experimentally to be correct. The LFV interactions have still been probed at many experiments such as the MEG experiment [64], the MEGA experiment [65], in the Belle detector at the KEKB [66], and BABAR detector at the PEP-II [67, 68]. In addition, at CERN there is a particle accelerator, which is currently under construction, namely the Large Hadron Collider (LHC), is expected to operate in May 2008 [69]. The collider tunnel in LHC consists of two pipes and each pipe contains a proton beam having an energy of 7.0 TeV which travels in opposite directions around the ring. This means that in total the collision energy will be 14 TeV. Furthermore, the International Linear Collider (ILC), which is also under construction, is planned to be completed in the late 2010s [69]. The initial collision energy is planned to be 500 GeV and this collision will be between electron and positron beams. It is expected that the physics beyond the SM will be detected at the LHC and ILC.

The simplest extension of the SM, the so called the 2HDM, possesses five physical Higgs bosons, namely, a charged pair (H^\pm), two neutral CP even scalars

(H^0 and h^0), and a neutral CP odd scalar (A^0). Observation of these particles in the experiments would become a clear signature for the physics beyond the SM.

Beside this phenomenological hint, the incompatibility of the predicted BRs of the LFV interactions with the experimental data obtained, the SM also possesses some conceptual problems which motivate us to look physics beyond, such as the hierarchy problem between EW and Planck scales. This problem could be achieved by introducing the extra dimensions. One possibility is proposed by Arkani-Hamed, Savas Dimopoulos and Gia Dvali [20, 21] with n compact extra spatial dimensions of large size. In this model, gravity spreads over all the volume including the extra dimensions while the matter fields are restricted in 4D brane. Another possibility is introduced by Randall and Sundrum [30, 31]. In this scenario, the extra dimension is compactified to S^1/Z_2 orbifold with two 4D brane boundaries. Even if the compactified extra dimensions are so small that we perceive the universe as 4D, if there are extra dimensions, fingerprints of them sure to exist and such fingerprints are the particles called KK particles which are the additional ingredients of the universe with extra dimensions. The masses of KK particles are determined by the higher dimensional geometry. In the large extra dimensions, for example, the masses of the KK particles are proportional to the inverse size of the extra dimension (for two extra dimensions, the size of the extra dimension is ~ 1 mm). This tells us that, the current and future accelerators should be able to discover them. Since in large extra dimensions, the only particle that can travel along the extra dimensions is called as graviton and thus, it is the only particle that has KK partners. However, the KK partners of the graviton interacts as weakly as the graviton itself. Therefore, KK partners of graviton would be hard to observe. In warped extra dimensions, however, we cannot take just the the inverse size of the extra dimension as the masses of KK particles which gives us the Planck scale mass and we know that on the TeV brane, nothing much heavier than a TeV can exist. If one calculate the masses of KK particles taking into account the warped space-time, the KK particles of graviton turn out to have masses of about a TeV and these KK particles, not as in the case of large extra dimensions, will interact sixteen orders of magnitude larger than the graviton itself in this geometry. This is good news, since depend-

ing on the ultimate energy reach in LHC, there is a probability of finding the KK partners of the graviton. Apart from the graviton, in the Randall Sundrum scenario, there exists an additional scalar field, that lives in the 5D bulk, such that the size of extra dimension is proportional to its vacuum expectation value and its fluctuation over this expectation value is called as the radion field. In order to avoid the conflict with the equivalence principle, the introduced field should be massive and to stabilize the distance between the branes, a potential for this field is proposed by Goldberger and Wise [50]. The radion decays are interesting since a considerable information about the scenario under consideration (the RS1 scenario) can be obtained with the help of accurate measurements.

In the present work we study the possible LFV decays of the radion field r and predict the BRs of the LFV r decays $r \rightarrow l_1^- l_2^+$ in the RS1 model. We observe that the BRs of the processes we study are at most of the order of 10^{-8} , for the small values of radion mass m_r and their sensitivities to m_r decrease with the increasing values of m_r . On the other hand, the BRs for $r \rightarrow \tau^\pm e^\pm$ ($r \rightarrow \mu^\pm e^\pm$) decays are of the order of 10^{-12} (10^{-15}) for the small values of m_r . These results show that, among these processes, the LFV $r \rightarrow \tau^\pm \mu^\pm$ decay would be the most appropriate one to measure its BR.

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APPENDIX A

The gauge invariance is an important concept in modern particle theories as it is the origin of all of the known four fundamental forces described in Chapter 2. The basic method to provide gauge invariance is to ensure that Lagrangian remains invariant under certain symmetry transformations which reflect conservation laws in nature. By applying these transformations, we end up with conserved physical quantities. Since these conserved quantities should not depend on position in space-time, theories of particle interactions have to be invariant under local as well as global gauge transformations explained below. The transformations could be written as

$$\psi \rightarrow U\psi, \tag{A-1}$$

where U is unitarity. So, the symmetry is called $U(1)$ gauge invariance.

Global Gauge Transformations

The expression for global gauge transformation (GGT) is

$$\psi \rightarrow e^{i\theta}\psi, \tag{A-2}$$

where θ is a real number. Thus, GGT represents an identical operation at all points in space-time and causes a simple shift in the phase of a fermion wave function. As a first step we can see how it works in QED. Equation of motion for free fermions are obtained from Dirac Lagrangian

$$L_{free} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi, \tag{A-3}$$

where ψ is the fermion spinor. It is clear that such a Lagrangian remains invariant under GGT. So, the global phase symmetry is just a statement of the fact that

the laws of physics are independent of the choice of phase convention.

Local Gauge Transformations

The expression for local gauge transformation (LGT) is

$$\psi \rightarrow e^{iq\theta(x)}\psi, \quad (\text{A-4})$$

where θ is a function of $x = (\mathbf{x}, t)$. Thus, LGT corresponds to choosing a convention to define the phase of the fermion wavefunction, which is different at different points in space-time. Going back to the Dirac Lagrangian defined in eq. A-3, one can simply realize that it is not invariant under this more demanding symmetry transformation. However, it comes as a pleasant surprise that if we introduce another field, A_μ , a Lagrangian which exhibits local gauge symmetry can be obtained. The required field must have infinite range, since there is no limit to the distances over which the phase transformations done. Hence, invariance of Lagrangian requires this new field to be massless. In fact, this field is not other than the long-range electromagnetic field: the photon. Therefore, we should modify our Lagrangian to make it gauge invariant by replacing the normal derivative ∂_μ with the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$. So, the Lagrangian in eq. A-3 reads

$$L = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - qA_\mu \bar{\psi}\gamma^\mu \psi = L_{free} - J^\mu A_\mu. \quad (\text{A-5})$$

It so happens that, the photon transforms under LGT as $A_\mu \rightarrow A_\mu + \partial_\mu \theta(x)$ so that, the changes in the Lagrangian resulting from the LGT is cancelled out by the changes in A_μ .

To see the complete picture, we should also add the gauge invariant kinetic term to the QED Lagrangian

$$L_{QED} = L_{free} - J^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad (\text{A-6})$$

where $F_{\mu\nu}$ is the electromagnetic field tensor defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (\text{A-7})$$

This Lagrangian expresses the interaction of Dirac fields with the massless photon.

APPENDIX B

The Einstein Equation in Another Form

Here we show another form of the Einstein equation which we use in the text. By using the equation $G_{MN} = R_{MN} - \frac{1}{2}g_{MN}R = \kappa^2 T_{MN}$ we obtain R_{MN} as

$$R_{MN} = \kappa^2 T_{MN} + \frac{1}{2}g_{MN}R, \quad (\text{B-1})$$

and multiplying this equation by g^{MN} from left we get

$$g^{MN}R_{MN} = \kappa^2 g^{MN}T_{MN} + \frac{1}{2}g^{MN}g_{MN}R, \quad (\text{B-2})$$

which is equal to,

$$R = \kappa^2 T + \frac{1}{2}dR = \frac{2\kappa^2}{2-d}T, \quad (\text{B-3})$$

where d is the number of dimensions. Substituting this into the eq. B-1 we obtain R_{MN} as

$$\begin{aligned} R_{MN} &= \kappa^2 T_{MN} + \frac{1}{2}g_{MN}\frac{2\kappa^2}{2-d}T \\ &= \kappa^2(T_{MN} + g_{MN}\frac{1}{2-d}T) \\ &= \kappa^2\tilde{T}_{MN}, \end{aligned} \quad (\text{B-4})$$

with

$$\tilde{T}_{MN} = T_{MN} - \frac{1}{3}g_{MN}T, \quad (\text{B-5})$$

where $d = 5$ in our case.

The Derivation of the Linearized Einstein Equations

Now, we consider the metric in eq. 4.110 and first we will derive the explicit form of the curvature R_{MN} and its linearized form. The curvature $R_{\mu\nu}$ in terms of connection coefficients can be written as

$$R_{\mu\nu} = \Gamma_{\mu K, \nu}^K - \Gamma_{\mu\nu, K}^K - \Gamma_{\mu\nu}^K \Gamma_{KM}^M + \Gamma_{\mu M}^K \Gamma_{\nu K}^M. \quad (\text{B-6})$$

Let us analyze this term by term. The first term is derived as follows

$$\begin{aligned} \Gamma_{\mu K, \nu}^K &= \{g^{KL} \Gamma_{L\mu K}\}_{, \nu} \\ &= \frac{1}{2} \{g^{KL} [g_{L\mu, K} + g_{LK, \mu} - g_{\mu K, L}]\}_{, \nu} \\ &= \frac{1}{2} \{g^{K\rho} [g_{\rho\mu, K} + g_{\rho K, \mu} - g_{\mu K, \rho}] + g^{K5} [g_{5K, \mu} - g_{\mu K, 5}]\}_{, \nu} \\ &= \frac{1}{2} \{g^{\xi\rho} [g_{\rho\mu, \xi} + g_{\rho\xi, \mu} - g_{\mu\xi, \rho}] + g^{55} [g_{55, \mu}]\}_{, \nu} \\ &= \frac{1}{2} \{e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu})_{, \xi} + (e^{-2A-2F} \eta_{\rho\xi})_{, \mu} - (e^{-2A-2F} \eta_{\mu\xi})_{, \rho}] \\ &\quad + (1+G)^{-2} (1+G)_{, \mu}^2\}_{, \nu} \\ &= \frac{1}{2} \{e^{2A+2F} \eta^{\xi\rho} [(-2e^{-2A-2F} \partial_\xi F \eta_{\rho\mu}) + (-2e^{-2A-2F} \partial_\mu F \eta_{\rho\xi}) \\ &\quad - (-2e^{-2A-2F} \partial_\rho F \eta_{\mu\xi}) + (1+G)^{-2} 2\partial_\mu G (1+G)]\}_{, \nu} \\ &= \{-\eta_\mu^\xi \partial_\xi F - 4\partial_\mu F + \eta_\mu^\rho \partial_\rho F + \frac{\partial_\mu G}{1+G}\}_{, \nu} \\ &= \{-\partial_\mu F - 4\partial_\mu F + \partial_\mu F + \frac{\partial_\mu G}{1+G}\}_{, \nu} \\ &= \{-4\partial_\mu F + \frac{\partial_\mu G}{1+G}\}_{, \nu} \\ &= -4\partial_\mu \partial_\nu F + \frac{\partial_\mu \partial_\nu G}{1+G} - \frac{\partial_\mu G \partial_\nu G}{(1+G)^2}, \end{aligned} \quad (\text{B-7})$$

and for small fluctuations we have

$$\Gamma_{\mu K, \nu}^K \cong -4\partial_\mu \partial_\nu F + \partial_\mu \partial_\nu G (1-G) - \partial_\mu G \partial_\nu G (1-2G). \quad (\text{B-8})$$

Then, the linearized form of the first term is obtained as

$$\delta\Gamma_{\mu K, \nu}^K = -4\partial_\mu\partial_\nu F + \partial_\mu\partial_\nu G. \quad (\text{B-9})$$

The second term is

$$\begin{aligned} \Gamma_{\mu\nu, K}^K &= \{g^{KL}\Gamma_{L\mu\nu}\}_{,K} \\ &= \frac{1}{2}\{g^{KL}[g_{L\mu, \nu} + g_{L\nu, \mu} - g_{\mu\nu, L}]\}_{,K} \\ &= \frac{1}{2}\{g^{K\rho}[g_{\rho\mu, \nu} + g_{\rho\nu, \mu} - g_{\mu\nu, \rho}] + g^{K5}[-g_{\mu\nu, 5}]\}_{,K} \\ &= \frac{1}{2}\{g^{\xi\rho}[g_{\rho\mu, \nu} + g_{\rho\nu, \mu} - g_{\mu\nu, \rho}]\}_{, \xi} + \frac{1}{2}\{g^{55}(-g_{\mu\nu, 5})\}_{, 5} \\ &= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}[(e^{-2A-2F}\eta_{\rho\mu})_{, \nu} + (e^{-2A-2F}\eta_{\rho\nu})_{, \mu} - (e^{-2A-2F}\eta_{\mu\nu})_{, \rho}]\}_{, \xi} \\ &\quad + \frac{1}{2}\{(1+G)^{-2}(e^{-2A-2F}\eta_{\mu\nu})_{, 5}\}_{, 5} \\ &= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}[(-2e^{-2A-2F}\partial_\nu F\eta_{\rho\mu}) + (-2e^{-2A-2F}\partial_\mu F\eta_{\rho\nu}) \\ &\quad - (-2e^{-2A-2F}\partial_\rho F\eta_{\mu\nu})]\}_{, \xi} + \frac{1}{2}\{-2(1+G)^{-2}\partial_5(A+F)e^{-2A-2F}\eta_{\mu\nu}\}_{, 5} \\ &= \{-\eta_\mu^\xi\partial_\nu F - \eta_\nu^\xi\partial_\mu F + \eta^{\xi\rho}\eta_{\mu\nu}\partial_\rho F\}_{, \xi} - \left\{\frac{e^{-2A-2F}\eta_{\mu\nu}\partial_5(A+F)}{(1+G)^2}\right\}_{, 5} \\ &= -\eta_\mu^\xi\partial_\xi\partial_\nu F - \eta_\nu^\xi\partial_\xi\partial_\mu F + \eta^{\xi\rho}\eta_{\mu\nu}\partial_\xi\partial_\rho F \\ &\quad - e^{-2A-2F}\eta_{\mu\nu}\left\{\frac{-2(A'+F')^2 + A'' + F''}{(1+G)^2} - \frac{2(A'+F')G'(1+G)}{(1+G)^4}\right\} \\ &= -\partial_\mu\partial_\nu F - \partial_\nu\partial_\mu F + \eta_{\mu\nu}\partial_\xi\partial^\xi F \\ &\quad - \frac{e^{-2A-2F}\eta_{\mu\nu}}{(1+G)^2}[-2(A'+F')^2 + A'' + F'' - \frac{2(A'+F')G'}{(1+G)}] \\ &= -2\partial_\mu\partial_\nu F + \eta_{\mu\nu}\square F - \frac{e^{-2A-2F}\eta_{\mu\nu}}{(1+G)^2} \\ &\quad \times [-2A'^2 - 2F'^2 - 4A'F' + A'' + F'' - \frac{2(A'+F')G'}{(1+G)}], \quad (\text{B-10}) \end{aligned}$$

and for small fluctuations we get

$$\begin{aligned}
\Gamma_{\mu K, \nu}^K &\cong -2\partial_\mu \partial_\nu F + \eta_{\mu\nu} \square F \\
&- e^{-2A} \eta_{\mu\nu} (1 - 2F)(1 - 2G) \\
&\times [A'' + F'' - 2A'^2 - 2F'^2 - 4A'F' - 2(A' + F')G'(1 - G)]. \text{(B-11)}
\end{aligned}$$

The linearized form of the second term is obtained as

$$\begin{aligned}
\delta\Gamma_{\mu\nu, K}^K &= -2\partial_\mu \partial_\nu F + \eta_{\mu\nu} \square F - e^{-2A} \eta_{\mu\nu} \\
&\times [F'' - 4A'F' - 2A'G' - 2FA'' + 4FA'^2 - 2GA'' + 4GA'^2]. \text{(B-12)}
\end{aligned}$$

The third term is calculated as

$$\begin{aligned}
\Gamma_{\mu\nu}^K \Gamma_{KM}^M &= g^{KL} \Gamma_{L\mu\nu} g^{MS} \Gamma_{SKM} \\
&= \frac{1}{2} g^{KL} [g_{L\mu,\nu} + g_{L\nu,\mu} - g_{\mu\nu,L}] \frac{1}{2} g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&= \frac{1}{4} g^{K\rho} [g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&+ \frac{1}{4} g^{K5} [-g_{\mu\nu,5}] g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] g^{MS} [g_{S\xi,M} + g_{SM,\xi} - g_{\xi M,S}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\nu,5}] g^{MS} [g_{S5,M} + g_{SM,5} - g_{5M,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] g^{\alpha S} [g_{S\xi,\alpha} + g_{S\alpha,\xi} - g_{\xi\alpha,S}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\nu,5}] g^{\alpha S} [g_{S5,\alpha} + g_{S\alpha,5}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] g^{5S} [g_{S\xi,5} + g_{S5,\xi}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\nu,5}] g^{5S} [g_{S5,5} + g_{S5,5} - g_{55,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] g^{\alpha\beta} [g_{\beta\xi,\alpha} + g_{\beta\alpha,\xi} - g_{\xi\alpha,\beta}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\nu,5}] g^{\alpha\beta} [g_{\beta\alpha,5}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}] g^{55} [g_{55,\xi}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\nu,5}] g^{55} [g_{55,5}] \\
&= \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu})_{,\nu} + (e^{-2A-2F} \eta_{\rho\nu})_{,\mu} - (e^{-2A-2F} \eta_{\mu\nu})_{,\rho}] \\
&\times e^{2A+2F} \eta^{\alpha\beta} [(e^{-2A-2F} \eta_{\beta\xi})_{,\alpha} + (e^{-2A-2F} \eta_{\beta\alpha})_{,\xi} - (e^{-2A-2F} \eta_{\xi\alpha})_{,\beta}] \\
&+ \frac{1}{4} (1+G)^{-2} (e^{-2A-2F} \eta_{\mu\nu})_{,5} e^{2A+2F} \eta^{\alpha\beta} (e^{-2A-2F} \eta_{\beta\alpha})_{,5} \\
&+ \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu})_{,\nu} + (e^{-2A-2F} \eta_{\rho\nu})_{,\mu} - (e^{-2A-2F} \eta_{\mu\nu})_{,\rho}] \\
&\times [(1+G)^{-2} (1+G)_{,\xi}^2] \\
&+ \frac{1}{4} (1+G)^{-2} [(e^{-2A-2F} \eta_{\mu\nu})_{,5}] (1+G)^{-2} (1+G)_{,5}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}[-2\eta_\mu^\xi \partial_\nu F - 2\eta_\nu^\xi \partial_\mu F + 2\eta_{\mu\nu} \partial^\xi F] \times [-2\eta_\xi^\alpha \partial_\alpha F - 8\partial_\xi F + 2\eta_\xi^\beta \partial_\beta F] \\
&+ \frac{1}{4}\left[\frac{16e^{-2A-2F}\eta_{\mu\nu}\partial_5(A+F)\partial_5(A+F)}{(1+G)^2}\right] \\
&+ \frac{1}{4}[-2\eta_\mu^\xi \partial_\nu F - 2\eta_\nu^\xi \partial_\mu F + 2\eta_{\mu\nu} \partial^\xi F]\left[2\frac{\partial_\xi G}{1+G}\right] \\
&+ \frac{1}{4}\left[-4\frac{e^{-2A-2F}\eta_{\mu\nu}\partial_5(A+F)}{(1+G)^2}\frac{\partial_5 G}{1+G}\right] \\
&= 8\partial_\mu F \partial_\nu F - 4\eta_{\mu\nu} \partial_\xi F \partial^\xi F + \frac{4e^{-2A-2F}\eta_{\mu\nu}(A'+F')^2}{(1+G)^2} \\
&- \frac{\partial_\mu G \partial_\nu F}{1+G} - \frac{\partial_\mu F \partial_\nu G}{1+G} + \frac{\eta_{\mu\nu} \partial^\xi F \partial_\xi G}{1+G} - \frac{e^{-2A-2F}\eta_{\mu\nu}(A'+F')G'}{(1+G)^3} \\
&= 8\partial_\mu F \partial_\nu F - \frac{\partial_\mu G \partial_\nu F}{1+G} - \frac{\partial_\mu F \partial_\nu G}{1+G} - \frac{e^{-2A-2F}\eta_{\mu\nu}}{(1+G)^2} \\
&\times \left[-4(A'+F')^2 + \frac{(A'+F')G'}{1+G}\right] + \eta_{\mu\nu}[-4(\partial F)^2 + \frac{\partial^\xi F \partial_\xi G}{1+G}], \tag{B-13}
\end{aligned}$$

and for small fluctuations we have

$$\begin{aligned}
\Gamma_{\mu\nu}^K \Gamma_{KM}^M &\cong 8\partial_\mu F \partial_\nu F - \partial_\mu G \partial_\nu F(1-G) - \partial_\mu F \partial_\nu G(1-G) \\
&- e^{-2A}\eta_{\mu\nu}(1-2F)(1-2G) \\
&\times [-4A'^2 - 4F'^2 - 8A'F' + (A'+F')G'(1-G)] \\
&+ \eta_{\mu\nu}[-4(\partial F)^2 + \partial^\xi F \partial_\xi G(1-G)]. \tag{B-14}
\end{aligned}$$

Then, the linearized form of the third term is obtained as

$$\delta(\Gamma_{\mu\nu}^K \Gamma_{KM}^M) = -e^{-2A}\eta_{\mu\nu}[-8A'F' + A'G' + 8FA'^2 + 8GA'^2]. \tag{B-15}$$

Finally, we will derive the fourth term in the same manner as

$$\begin{aligned}
\Gamma_{\mu M}^K \Gamma_{\nu K}^M &= g^{KL} \Gamma_{L\mu M} g^{MS} \Gamma_{S\nu K} \\
&= \frac{1}{2} g^{KL} [g_{L\mu, M} + g_{LM, \mu} - g_{\mu M, L}] \frac{1}{2} g^{MS} [g_{S\nu, K} + g_{SK, \nu} - g_{\nu K, S}] \\
&= \frac{1}{4} g^{K\rho} [g_{\rho\mu, M} + g_{\rho M, \mu} - g_{\mu M, \rho}] g^{MS} [g_{S\nu, K} + g_{SK, \nu} - g_{\nu K, S}] \\
&+ \frac{1}{4} g^{K5} [g_{5M, \mu} - g_{\mu M, 5}] g^{MS} [g_{S\nu, K} + g_{SK, \nu} - g_{\nu K, S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, M} + g_{\rho M, \mu} - g_{\mu M, \rho}] g^{MS} [g_{S\nu, \xi} + g_{S\xi, \nu} - g_{\nu\xi, S}] \\
&+ \frac{1}{4} g^{55} [g_{5M, \mu} - g_{\mu M, 5}] g^{MS} [g_{S\nu, 5} + g_{S5, \nu}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, \alpha} + g_{\rho\alpha, \mu} - g_{\mu\alpha, \rho}] g^{\alpha S} [g_{S\nu, \xi} + g_{S\xi, \nu} - g_{\nu\xi, S}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\alpha, 5}] g^{\alpha S} [g_{S\nu, 5} + g_{S5, \nu}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, 5}] g^{5S} [g_{S\nu, \xi} + g_{S\xi, \nu} - g_{\nu\xi, S}] \\
&+ \frac{1}{4} g^{55} [g_{55, \mu}] g^{5S} [g_{S\nu, 5} + g_{S5, \nu}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, \alpha} + g_{\rho\alpha, \mu} - g_{\mu\alpha, \rho}] g^{\alpha\beta} [g_{\beta\nu, \xi} + g_{\beta\xi, \nu} - g_{\nu\xi, \beta}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\alpha, 5}] g^{\alpha\beta} [g_{\beta\nu, 5}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, 5}] g^{55} [-g_{\nu\xi, 5}] \\
&+ \frac{1}{4} g^{55} [g_{55, \mu}] g^{55} [g_{55, \nu}] \\
&= \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu})_{, \alpha} + (e^{-2A-2F} \eta_{\rho\alpha})_{, \mu} - (e^{-2A-2F} \eta_{\mu\alpha})_{, \rho}] \\
&\times e^{2A+2F} \eta^{\alpha\beta} [(e^{-2A-2F} \eta_{\beta\nu})_{, \xi} + (e^{-2A-2F} \eta_{\beta\xi})_{, \nu} - (e^{-2A-2F} \eta_{\nu\xi})_{, \beta}] \\
&+ \frac{1}{4} (1+G)^{-2} (e^{-2A-2F} \eta_{\mu\alpha})_{, 5} e^{2A+2F} \eta^{\alpha\beta} (e^{-2A-2F} \eta_{\beta\nu})_{, 5} \\
&+ \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} (e^{-2A-2F} \eta_{\rho\mu})_{, 5} (1+G)^{-2} (e^{-2A-2F} \eta_{\nu\xi})_{, 5} \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{, \mu}^2 (1+G)^{-2} (1+G)_{, \nu}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}[-2\eta_\mu^\xi \partial_\alpha F - 2\eta_\alpha^\xi \partial_\mu F + 2\eta_{\mu\alpha} \partial^\xi F] \times [-2\eta_\nu^\alpha \partial_\xi F - 2\eta_\xi^\alpha \partial_\nu F + 2\eta_{\nu\xi} \partial^\alpha F] \\
&+ \frac{1}{4} \left[\frac{4e^{-2A-2F} \eta_{\mu\alpha} \eta_\nu^\alpha \partial_5(A+F) \partial_5(A+F)}{(1+G)^2} \right] \\
&+ \frac{1}{4} \left[\frac{4e^{-2A-2F} \eta_\mu^\xi \eta_{\nu\xi} \partial_5(A+F) \partial_5(A+F)}{(1+G)^2} \right] \\
&+ \frac{1}{4} \left[\frac{4\partial_\mu G \partial_\nu G}{(1+G)^2} \right] \\
&= 6\partial_\mu F \partial_\nu F - 2\eta_{\mu\nu} (\partial F)^2 + \frac{2\eta_{\mu\nu} e^{-2A-2F} (A' + F')^2}{(1+G)^2} \\
&+ \frac{\partial_\mu G \partial_\nu G}{(1+G)^2}. \tag{B-16}
\end{aligned}$$

Since we have small fluctuations, we can write

$$\begin{aligned}
\Gamma_{\mu M}^K \Gamma_{\nu K}^M &\cong 6\partial_\mu F \partial_\nu F - 2\eta_{\mu\nu} (\partial F)^2 + \eta_{\mu\nu} e^{-2A} (1-2F)(1-2G) \\
&\times [2A'^2 + 2F'^2 + 4A'F'] + \partial_\mu G \partial_\nu G (1-2G). \tag{B-17}
\end{aligned}$$

As a result, the linearized form of the fourth term is obtained as

$$\delta(\Gamma_{\mu M}^K \Gamma_{\nu K}^M) = \eta_{\mu\nu} e^{-2A} [4A'F' - 4A'^2 F - 4A'^2 G]. \tag{B-18}$$

If we add these four terms, we get $\delta R_{\mu\nu}$ as

$$\begin{aligned}
\delta R_{\mu\nu} &= -2\partial_\mu \partial_\nu F + \partial_\mu \partial_\nu G - \eta_{\mu\nu} \square F + e^{-2A} \eta_{\mu\nu} \\
&\times [F'' - 4A'F' - 2A'G' - 2FA'' + 4FA'^2 - 2GA'' + 4GA'^2 \\
&- 8A'F' + A'G' + 8FA'^2 + 8GA'^2 + 4A'F' - 4A'^2 F - 4A'^2 G] \\
&= -2\partial_\mu \partial_\nu F + \partial_\mu \partial_\nu G - \eta_{\mu\nu} \square F + e^{-2A} \eta_{\mu\nu} \\
&\times [F'' - 8A'F' - A'G' - 2FA'' + 8FA'^2 - 2GA'' + 8GA'^2]. \tag{B-19}
\end{aligned}$$

Now, we will calculate the curvature $R_{\mu 5}$ which can be written in terms of connection coefficients as

$$R_{\mu 5} = \Gamma_{\mu K, 5}^K - \Gamma_{\mu 5, K}^K - \Gamma_{\mu 5}^K \Gamma_{KM}^M + \Gamma_{\mu M}^K \Gamma_{5K}^M. \tag{B-20}$$

Let us analyze the $R_{\mu 5}$ also term by term. The first term is obtained as

$$\begin{aligned}
\Gamma_{\mu K,5}^K &= \{g^{KL}\Gamma_{L\mu K}\}_{,5} \\
&= \frac{1}{2}\{g^{KL}[g_{L\mu,K} + g_{LK,\mu} - g_{\mu K,L}]\}_{,5} \\
&= \frac{1}{2}\{g^{K\rho}[g_{\rho\mu,K} + g_{\rho K,\mu} - g_{\mu K,\rho}] + g^{K5}[g_{5K,\mu} - g_{\mu K,5}]\}_{,5} \\
&= \frac{1}{2}\{g^{\xi\rho}[g_{\rho\mu,\xi} + g_{\rho\xi,\mu} - g_{\mu\xi,\rho}] + g^{55}[g_{55,\mu}]\}_{,5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}[(e^{-2A-2F}\eta_{\rho\mu})_{,\xi} + (e^{-2A-2F}\eta_{\rho\xi})_{,\mu} - (e^{-2A-2F}\eta_{\mu\xi})_{,\rho}] \\
&\quad + (1+G)^{-2}(1+G)_{,\mu}^2\}_{,5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}[(-2e^{-2A-2F}\partial_\xi F\eta_{\rho\mu}) + (-2e^{-2A-2F}\partial_\mu F\eta_{\rho\xi}) \\
&\quad - (-2e^{-2A-2F}\partial_\rho F\eta_{\mu\xi}) + (1+G)^{-2}2\partial_\mu G(1+G)]\}_{,5} \\
&= \{-\eta_\mu^\xi\partial_\xi F - 4\partial_\mu F + \eta_\mu^\rho\partial_\rho F + \frac{\partial_\mu G}{1+G}\}_{,5} \\
&= \{-\partial_\mu F - 4\partial_\mu F + \partial_\mu F + \frac{\partial_\mu G}{1+G}\}_{,5} \\
&= \{-4\partial_\mu F + \frac{\partial_\mu G}{1+G}\}_{,5} \\
&= -4\partial_\mu F' + \frac{\partial_\mu G'}{1+G} - \frac{G'\partial_\mu G}{(1+G)^2}.
\end{aligned} \tag{B-21}$$

Since $G = 2F$ we get

$$\Gamma_{\mu K,5}^K = -4\partial_\mu F' + \frac{2\partial_\mu F'}{1+2F} - \frac{4F'\partial_\mu F}{(1+2F)^2}, \tag{B-22}$$

and taking into account the small fluctuation we obtain

$$\Gamma_{\mu K,5}^K \cong -4\partial_\mu F' + 2\partial_\mu F'(1-2F) - 4F'\partial_\mu F(1-4F). \tag{B-23}$$

Then, the linearized form of the first term will be

$$\begin{aligned}
\delta\Gamma_{\mu K,5}^K &= -4\partial_\mu F' + 2\partial_\mu F' \\
&= -2\partial_\mu F'.
\end{aligned} \tag{B-24}$$

The second term can be calculated as follows

$$\begin{aligned}
\Gamma_{\mu 5, K}^K &= \{g^{KL}\Gamma_{L\mu 5}\}_{,K} \\
&= \frac{1}{2}\{g^{KL}[g_{L\mu, 5} + g_{L5, \mu}]\}_{,K} \\
&= \frac{1}{2}\{g^{K\rho}[g_{\rho\mu, 5}] + g^{K5}[g_{55, \mu}]\}_{,K} \\
&= \frac{1}{2}\{g^{\xi\rho}[g_{\rho\mu, 5}]\}_{, \xi} + \frac{1}{2}\{g^{55}(g_{55, \mu})\}_{, 5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}[(e^{-2A-2F}\eta_{\rho\mu})_{, 5}]\}_{, \xi} \\
&\quad + \frac{1}{2}\{(1+G)^{-2}(1+G)_{, \mu}^2\}_{, 5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}[(-2e^{-2A-2F}\partial_5(A+F)\eta_{\rho\mu})]\}_{, \xi} \\
&\quad + \frac{1}{2}\{(1+G)^{-2}2(1+G)\partial_\mu G\}_{, 5} \\
&= \{-\eta_\mu^\xi\partial_5(A+F)\}_{, \xi} + \{\frac{\partial_\mu G}{(1+G)}\}_{, 5} \\
&= -\partial_\mu F' + \frac{\partial_\mu G'}{(1+G)} - \frac{G'\partial_\mu G}{(1+G)^2}.
\end{aligned} \tag{B-25}$$

Substituting $G = 2F$ we have

$$\Gamma_{\mu 5, K}^K = -\partial_\mu F' + \frac{2\partial_\mu F'}{(1+2F)} - \frac{4F'\partial_\mu F}{(1+2F)^2}, \tag{B-26}$$

and due to the small fluctuation in F it can be written as

$$\Gamma_{\mu 5, K}^K \cong -\partial_\mu F' + 2\partial_\mu F'(1-2F) - 4F'\partial_\mu F(1-4F). \tag{B-27}$$

We get the linearized form of the second term as

$$\begin{aligned}
\delta\Gamma_{\mu 5, K}^K &= -\partial_\mu F' + 2\partial_\mu F' \\
&= \partial_\mu F'.
\end{aligned} \tag{B-28}$$

The third term is

$$\begin{aligned}
\Gamma_{\mu 5}^K \Gamma_{KM}^M &= g^{KL} \Gamma_{L\mu 5} g^{MS} \Gamma_{SKM} \\
&= \frac{1}{2} g^{KL} [g_{L\mu,5} + g_{L5,\mu}] \frac{1}{2} g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&= \frac{1}{4} g^{K\rho} [g_{\rho\mu,5}] g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&+ \frac{1}{4} g^{K5} [g_{55,\mu}] g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,5}] g^{MS} [g_{S\xi,M} + g_{SM,\xi} - g_{\xi M,S}] \\
&+ \frac{1}{4} g^{55} [g_{55,\mu}] g^{MS} [g_{S5,M} + g_{SM,5} - g_{5M,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,5}] g^{\alpha S} [g_{S\xi,\alpha} + g_{S\alpha,\xi} - g_{\xi\alpha,S}] \\
&+ \frac{1}{4} g^{55} [g_{55,\mu}] g^{\alpha S} [g_{S5,\alpha} + g_{S\alpha,5}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,5}] g^{5S} [g_{S\xi,5} + g_{S5,\xi}] \\
&+ \frac{1}{4} g^{55} [g_{55,\mu}] g^{5S} [g_{S5,5} + g_{S5,5} - g_{55,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,5}] g^{\alpha\beta} [g_{\beta\xi,\alpha} + g_{\beta\alpha,\xi} - g_{\xi\alpha,\beta}] \\
&+ \frac{1}{4} g^{55} [g_{55,\mu}] g^{\alpha\beta} [g_{\beta\alpha,5}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu,5}] g^{55} [g_{55,\xi}] \\
&+ \frac{1}{4} g^{55} [g_{55,\mu}] g^{55} [g_{55,5}] \\
&= \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu}),_5] \\
&\times e^{2A+2F} \eta^{\alpha\beta} [(e^{-2A-2F} \eta_{\beta\xi}),_\alpha + (e^{-2A-2F} \eta_{\beta\alpha}),_\xi - (e^{-2A-2F} \eta_{\xi\alpha}),_\beta] \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{,\mu}^2 e^{2A+2F} \eta^{\alpha\beta} (e^{-2A-2F} \eta_{\beta\alpha}),_5 \\
&+ \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu}),_5 (1+G)^{-2} (1+G)_{,\xi}^2] \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{,\mu}^2 (1+G)^{-2} (1+G)_{,5}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}[-2\eta_\mu^\xi \partial_5(A+F)] \times [-2\eta_\xi^\alpha \partial_\alpha F - 8\partial_\xi F + 2\eta_\xi^\beta \partial_\beta F] \\
&+ \frac{1}{4}\left[\frac{-16\partial_\mu G \partial_5(A+F)}{(1+G)}\right] \\
&+ \frac{1}{4}\left[-4\eta_\mu^\xi \partial_5(A+F) \frac{\partial_\xi G}{1+G}\right] \\
&+ \frac{1}{4}\left[4\frac{\partial_\mu G}{(1+G)} \frac{\partial_5 G}{1+G}\right] \\
&= 4\partial_\mu F(A'+F') - \frac{4\partial_\mu G(A'+F')}{1+G} \\
&- \frac{\partial_\mu G(A'+F')}{1+G} + \frac{G'\partial_\mu G}{(1+G)^2}.
\end{aligned} \tag{B-29}$$

By using $G = 2F$ we get

$$\Gamma_{\mu 5}^K \Gamma_{KM}^M = 4\partial_\mu F(A'+F') - \frac{10\partial_\mu F(A'+F')}{1+2F} + \frac{4F'\partial_\mu F}{(1+2F)^2}, \tag{B-30}$$

and for small fluctuations we have

$$\begin{aligned}
\Gamma_{\mu 5}^K \Gamma_{KM}^M &\cong 4\partial_\mu F(A'+F') - 10\partial_\mu F(A'+F')(1-2F) \\
&+ 4F'\partial_\mu F(1-4F).
\end{aligned} \tag{B-31}$$

Then, the third term in the linearized form is obtained as

$$\begin{aligned}
\delta(\Gamma_{\mu 5}^K \Gamma_{KM}^M) &= 4A'\partial_\mu F - 10A'\partial_\mu F \\
&= -6A'\partial_\mu F.
\end{aligned} \tag{B-32}$$

The last term is calculated as

$$\begin{aligned}
\Gamma_{\mu M}^K \Gamma_{5K}^M &= g^{KL} \Gamma_{L\mu M} g^{MS} \Gamma_{S5K} \\
&= \frac{1}{2} g^{KL} [g_{L\mu, M} + g_{LM, \mu} - g_{\mu M, L}] \frac{1}{2} g^{MS} [g_{S5, K} + g_{SK, 5} - g_{5K, S}] \\
&= \frac{1}{4} g^{K\rho} [g_{\rho\mu, M} + g_{\rho M, \mu} - g_{\mu M, \rho}] g^{MS} [g_{S5, K} + g_{SK, 5} - g_{5K, S}] \\
&+ \frac{1}{4} g^{K5} [g_{5M, \mu} - g_{\mu M, 5}] g^{MS} [g_{S5, K} + g_{SK, 5} - g_{5K, S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, M} + g_{\rho M, \mu} - g_{\mu M, \rho}] g^{MS} [g_{S5, \xi} + g_{S\xi, 5} \\
&+ \frac{1}{4} g^{55} [g_{5M, \mu} - g_{\mu M, 5}] g^{MS} [g_{S5, 5} + g_{S5, 5} - g_{55, S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, \alpha} + g_{\rho\alpha, \mu} - g_{\mu\alpha, \rho}] g^{\alpha S} [g_{S5, \xi} + g_{S\xi, 5}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\alpha, 5}] g^{\alpha S} [2g_{S5, 5} - g_{55, S}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, 5}] g^{5S} [g_{S5, \xi} + g_{S\xi, 5}] \\
&+ \frac{1}{4} g^{55} [g_{55, \mu}] g^{5S} [2g_{S5, 5} - g_{55, S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, \alpha} + g_{\rho\alpha, \mu} - g_{\mu\alpha, \rho}] g^{\alpha\beta} [g_{\beta\xi, 5}] \\
&+ \frac{1}{4} g^{55} [-g_{\mu\alpha, 5}] g^{\alpha\beta} [-g_{55, \beta}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{\rho\mu, 5}] g^{55} [g_{55, \xi}] \\
&+ \frac{1}{4} g^{55} [g_{55, \mu}] g^{55} [g_{55, 5}] \\
&= \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\mu})_{, \alpha} + (e^{-2A-2F} \eta_{\rho\alpha})_{, \mu} - (e^{-2A-2F} \eta_{\mu\alpha})_{, \rho}] \\
&\times e^{2A+2F} \eta^{\alpha\beta} [(e^{-2A-2F} \eta_{\beta\xi})_{, 5}] \\
&+ \frac{1}{4} (1+G)^{-2} (e^{-2A-2F} \eta_{\mu\alpha})_{, 5} e^{2A+2F} \eta^{\alpha\beta} (1+G)_{, \beta}^2 \\
&+ \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} (e^{-2A-2F} \eta_{\rho\mu})_{, 5} (1+G)^{-2} (1+G)_{, \xi}^2 \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{, \mu}^2 (1+G)^{-2} (1+G)_{, 5}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}[-2\eta_\mu^\xi \partial_\alpha F + 2\eta_\alpha^\xi \partial_\mu F - 2\eta_{\mu\alpha} \partial^\xi F] \times [-2\eta_\xi^\alpha \partial_5(A+F)] \\
&+ \frac{1}{4}\left[\frac{-4e^{-2A-2F}\eta_{\mu\alpha}e^{2A+2F}\eta^{\alpha\beta}\partial_5(A+F)\partial_\beta G}{(1+G)}\right] \\
&+ \frac{1}{4}\left[\frac{-4\eta_\mu^\xi \partial_5(A+F)\partial_\xi G}{1+G} + \frac{4\partial_\mu G \partial_5 G}{(1+G)^2}\right] \\
&= 4\partial_\mu F(A'+F') - \frac{2(A'+F')\partial_\mu G}{1+G} + \frac{G'\partial_\mu G}{(1+G)^2}.
\end{aligned} \tag{B-33}$$

Substituting $G = 2F$, for small fluctuations, we get

$$\begin{aligned}
\Gamma_{\mu M}^K \Gamma_{5K}^M &= 4\partial_\mu F(A'+F') - \frac{4(A'+F')\partial_\mu F}{1+2F} + \frac{4F'\partial_\mu F}{(1+2F)^2} \\
&\cong 4\partial_\mu F(A'+F') - 4(A'+F')\partial_\mu F(1-2F) \\
&+ 4F'\partial_\mu F(1-4F).
\end{aligned} \tag{B-34}$$

We obtain the linearized form of the fourth term as

$$\begin{aligned}
\delta(\Gamma_{\mu M}^K \Gamma_{5K}^M) &= 4A'\partial_\mu F - 4A'\partial_\mu F \\
&= 0.
\end{aligned} \tag{B-35}$$

If we add these four terms, the linearized form of $R_{\mu 5}$ reads

$$\begin{aligned}
\delta R_{\mu 5} &= -2\partial_\mu F' - \partial_\mu F' + 6A'\partial_\mu F + 0 \\
&= -3\partial_\mu F' + 6A'\partial_\mu F.
\end{aligned} \tag{B-36}$$

Finally, we will derive the curvature R_{55} and, in terms of connection coefficients, it reads

$$R_{55} = \Gamma_{5K,5}^K - \Gamma_{55,K}^K - \Gamma_{55}^K \Gamma_{KM}^M + \Gamma_{5M}^K \Gamma_{5K}^M. \tag{B-37}$$

Now, we start with the first term in R_{55} which reads

$$\begin{aligned}
\Gamma_{5K,5}^K &= \{g^{KL}\Gamma_{L5K}\}_{,5} \\
&= \frac{1}{2}\{g^{KL}[g_{L5,K} + g_{LK,5} - g_{5K,L}]\}_{,5} \\
&= \frac{1}{2}\{g^{K\rho}[g_{\rho K,5} - g_{5K,\rho}] + g^{K5}[g_{55,K} + g_{5K,5} - g_{5K,5}]\}_{,5} \\
&= \frac{1}{2}\{g^{\xi\rho}[g_{\rho\xi,5}] + g^{55}[g_{55,5}]\}_{,5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}(e^{-2A-2F}\eta_{\rho\xi})_{,5} + (1+G)^{-2}(1+G)_5^2\}_{,5} \\
&= \frac{1}{2}\{-8\partial_5(A+F) + \frac{2\partial_5 G}{1+G}\}_{,5} \\
&= \{-4(A' + F') + \frac{G'}{1+G}\}_{,5} \\
&= -4(A'' + F'') + \frac{G''}{1+G} - \frac{G'^2}{(1+G)^2}.
\end{aligned} \tag{B-38}$$

Since $G = 2F$ we get

$$\Gamma_{5K,5}^K = -4(A'' + F'') + \frac{2F''}{1+2F} - \frac{4F'^2}{(1+2F)^2}. \tag{B-39}$$

For small fluctuations we have

$$\Gamma_{5K,5}^K \cong -4(A'' + F'') + 2F''(1-2F) - 4F'^2(1-4F). \tag{B-40}$$

Then, one can simply write the linearized form of the first term as

$$\begin{aligned}
\delta\Gamma_{5K,5}^K &= -4F'' + 2F'' \\
&= -2F''.
\end{aligned} \tag{B-41}$$

The second term is

$$\begin{aligned}
\Gamma_{55,K}^K &= \{g^{KL}\Gamma_{L55}\}_{,K} \\
&= \frac{1}{2}\{g^{KL}[g_{L5,5} + g_{L5,5} - g_{55,L}]\}_{,K} \\
&= \frac{1}{2}\{g^{K\rho}[-g_{55,\rho}] + g^{K5}[2g_{55,5} - g_{55,5}]\}_{,K} \\
&= \frac{1}{2}\{g^{\xi\rho}[-g_{55,\rho}]\}_{,\xi} + \frac{1}{2}\{g^{55}(g_{55,5})\}_{,5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}(1+G)^2_{,\rho}\}_{,\xi} + \frac{1}{2}\{(1+G)^{-2}(1+G)^2_{,5}\}_{,5} \\
&= \frac{1}{2}\{e^{2A+2F}\eta^{\xi\rho}2\partial_\rho G(1+G)\}_{,\xi} + \frac{1}{2}\left\{\frac{G'}{1+G}\right\}_{,5} \\
&= e^{2A+2F}[2\partial_\xi F\partial^\xi G(1+G) + \partial_\xi\partial^\xi G(1+G) + \partial_\xi G\partial^\xi G] \\
&\quad + \frac{G''}{1+G} - \frac{G'^2}{(1+G)^2}.
\end{aligned} \tag{B-42}$$

Substituting $G = 2F$ we get

$$\begin{aligned}
\Gamma_{55,K}^K &= e^{2A+2F}[4(\partial F)^2(1+2F) + 2\Box F(1+2F) + 4(\partial F)^2] \\
&\quad + \frac{2F''}{1+2F} - \frac{4F'^2}{(1+2F)^2},
\end{aligned} \tag{B-43}$$

and for small fluctuations we have

$$\begin{aligned}
\Gamma_{55,K}^K &\cong e^{2A}(1+2F)[4(\partial F)^2(1+2F) + 2\Box F(1+2F) + 4(\partial F)^2] \\
&\quad + 2F''(1-2F) - 4F'^2(1-4F).
\end{aligned} \tag{B-44}$$

Then, the linearized form of the second term is obtained as

$$\delta\Gamma_{55,K}^K = 2e^{2A}\Box F + 2F''. \tag{B-45}$$

The third term in the curvature R_{55} is

$$\begin{aligned}
\Gamma_{55}^K \Gamma_{KM}^M &= g^{KL} \Gamma_{L55} g^{MS} \Gamma_{SKM} \\
&= \frac{1}{2} g^{KL} [g_{L5,5} + g_{L5,5} - g_{55,L}] \frac{1}{2} g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&= \frac{1}{4} g^{K\rho} [-g_{55,\rho}] g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&+ \frac{1}{4} g^{K5} [2g_{55,5} - g_{55,5}] g^{MS} [g_{SK,M} + g_{SM,K} - g_{KM,S}] \\
&= \frac{1}{4} g^{\xi\rho} [-g_{55,\rho}] g^{MS} [g_{S\xi,M} + g_{SM,\xi} - g_{\xi M,S}] \\
&+ \frac{1}{4} g^{55} [g_{55,5}] g^{MS} [g_{S5,M} + g_{SM,5} - g_{5M,S}] \\
&= \frac{1}{4} g^{\xi\rho} [-g_{55,\rho}] g^{\alpha S} [g_{S\xi,\alpha} + g_{S\alpha,\xi} - g_{\xi\alpha,S}] \\
&+ \frac{1}{4} g^{55} [g_{55,5}] g^{\alpha S} [g_{S5,\alpha} + g_{S\alpha,5}] \\
&+ \frac{1}{4} g^{\xi\rho} [-g_{55,\rho}] g^{5S} [g_{S\xi,5} + g_{S5,\xi}] \\
&+ \frac{1}{4} g^{55} [g_{55,5}] g^{5S} [g_{S5,5} + g_{S5,5} - g_{55,S}] \\
&= \frac{1}{4} g^{\xi\rho} [-g_{55,\rho}] g^{\alpha\beta} [g_{\beta\xi,\alpha} + g_{\beta\alpha,\xi} - g_{\xi\alpha,\beta}] \\
&+ \frac{1}{4} g^{55} [g_{55,5}] g^{\alpha\beta} [g_{\beta\alpha,5}] \\
&+ \frac{1}{4} g^{\xi\rho} [-g_{55,\rho}] g^{55} [g_{55,\xi}] \\
&+ \frac{1}{4} g^{55} [g_{55,5}] g^{55} [g_{55,5}] \\
&= \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} (1+G)_{,\rho}^2 e^{2A+2F} \eta^{\alpha\beta} \\
&\times [(e^{-2A-2F} \eta_{\beta\xi})_{,\alpha} + (e^{-2A-2F} \eta_{\beta\alpha})_{,\xi} - (e^{-2A-2F} \eta_{\xi\alpha})_{,\beta}] \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{,5}^2 e^{2A+2F} \eta^{\alpha\beta} (e^{-2A-2F} \eta_{\beta\alpha})_{,5} \\
&+ \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(1+G)_{,rho}^2 (1+G)^{-2} (1+G)_{,\xi}^2] \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{,5}^2 (1+G)^{-2} (1+G)_{,5}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}[e^{2A+2F}2(1+G)\partial^\xi G] \times [-2\eta_\xi^\alpha \partial_\alpha F - 8\partial_\xi F + 2\eta_\xi^\beta \partial_\beta F] \\
&+ \frac{1}{4}\left[\frac{-16G'\partial_5(A+F)}{(1+G)}\right] \\
&+ \frac{1}{4}[e^{2A+2F}2(1+G)\partial^\xi G \frac{2\partial_\xi G}{1+G}] \\
&+ \frac{1}{4}\left[4\frac{G'}{1+G}\frac{\partial_5 G'}{1+G}\right] \\
&= -4e^{2A+2F}(1+G)\partial^\xi G\partial_\xi F - \frac{4(A'+F')G'}{(1+G)} \\
&+ e^{2A+2F}\partial^\xi G\partial_\xi G + \frac{G'^2}{(1+G)^2}, \tag{B-46}
\end{aligned}$$

and by using $G = 2F$, the third term is obtained to be equal to

$$\begin{aligned}
\Gamma_{55}^K \Gamma_{KM}^M &= -8e^{2A+2F}(1+2F)(\partial F)^2 - \frac{8(A'+F')F'}{1+2F} \\
&+ 4e^{2A+2F}(\partial F)^2 + \frac{4F'^2}{(1+2F)^2}, \tag{B-47}
\end{aligned}$$

and, considering the small fluctuation, we get

$$\begin{aligned}
\Gamma_{55}^K \Gamma_{KM}^M &\cong -8e^{2A}(1+2F)(1+2F)(\partial F)^2 - 8(A'+F')F'(1-2F) \\
&+ 4e^{2A}(1+2F)(\partial F)^2 + 4F'^2(1-4F). \tag{B-48}
\end{aligned}$$

Then, the linearized form of the third term is obtained as

$$\delta(\Gamma_{55}^K \Gamma_{KM}^M) = -8A'F'. \tag{B-49}$$

Finally, the fourth term is derived as follows:

$$\begin{aligned}
\Gamma_{5M}^K \Gamma_{5K}^M &= g^{KL} \Gamma_{L5M} g^{MS} \Gamma_{S5K} \\
&= \frac{1}{2} g^{KL} [g_{L5,M} + g_{LM,5} - g_{5M,L}] \frac{1}{2} g^{MS} [g_{S5,K} + g_{SK,5} - g_{5K,S}] \\
&= \frac{1}{4} g^{K\rho} [g_{\rho M,5} - g_{5M,\rho}] g^{MS} [g_{S5,K} + g_{SK,5} - g_{5K,S}] \\
&+ \frac{1}{4} g^{K5} [g_{55,M}] g^{MS} [g_{S5,K} + g_{SK,5} - g_{5K,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho M,5} - g_{5M,\rho}] g^{MS} [g_{S5,\xi} + g_{S\xi,5}] \\
&+ \frac{1}{4} g^{55} [g_{55,M}] g^{MS} [g_{S5,5} + g_{55,5} - g_{55,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\alpha,5}] g^{\alpha S} [g_{S5,\xi} + g_{S\xi,5}] \\
&+ \frac{1}{4} g^{55} [g_{55,\alpha}] g^{\alpha S} [2g_{S5,5} - g_{55,S}] \\
&+ \frac{1}{4} g^{\xi\rho} [g_{55,\rho}] g^{5S} [g_{S5,\xi} + g_{S\xi,5}] \\
&+ \frac{1}{4} g^{55} [g_{55,M}] g^{5S} [2g_{S5,5} - g_{55,S}] \\
&= \frac{1}{4} g^{\xi\rho} [g_{\rho\alpha,5}] g^{\alpha\beta} [g_{\beta\xi,5}] \\
&+ \frac{1}{4} g^{55} [g_{55,\alpha}] g^{\alpha\beta} [-g_{55,\beta}] \\
&+ \frac{1}{4} g^{\xi\rho} [-g_{55,\rho}] g^{55} [g_{55,\xi}] \\
&+ \frac{1}{4} g^{55} [g_{55,5}] g^{55} [g_{55,5}] \\
&= \frac{1}{4} e^{2A+2F} \eta^{\xi\rho} [(e^{-2A-2F} \eta_{\rho\alpha})_{,5} e^{2A+2F} \eta^{\beta\alpha} (e^{-2A-2F} \eta_{\beta\xi})_{,5}] \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{,\alpha}^2 e^{2A+2F} \eta^{\alpha\beta} (1+G)_{,\beta}^2 \\
&+ \frac{1}{4} [e^{2A+2F} \eta^{\xi\rho} (1+G)_{,\rho}^2] [(1+G)^{-2} (1+G)_{,\xi}^2] \\
&+ \frac{1}{4} (1+G)^{-2} (1+G)_{,5}^2 (1+G)^{-2} (1+G)_{,5}^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}[-2\eta_\alpha^\xi \partial_5(A+F)][-2\eta_\xi^\alpha \partial_5(A+F)] \\
&+ \frac{1}{4}\left[\frac{4e^{2A+2F}\eta^{\alpha\beta}(1+G)\partial_\alpha G\partial_\beta G}{1+G}\right] \\
&+ \frac{1}{4}\left[\frac{4e^{2A+2F}\partial^\xi G\partial_\xi G(1+G)}{1+G}\right] \\
&+ \frac{1}{4}\left[4\frac{G'^2}{(1+G)^2}\right] \\
&= 4(A'+F')^2 + 2e^{2A+2F}(\partial G)^2 + \frac{G'^2}{(1+G)^2}. \tag{B-50}
\end{aligned}$$

Substituting $G = 2F$ we obtain

$$\Gamma_{5M}^K \Gamma_{5K}^M = 4(A'+F')^2 + 8e^{2A+2F}(\partial F)^2 + \frac{4F'^2}{(1+2F)^2}, \tag{B-51}$$

and, with small fluctuations, we have

$$\Gamma_{5M}^K \Gamma_{5K}^M \cong 4(A'+F')^2 + 8e^{2A}(1+2F)(\partial F)^2 + 4F'^2(1-4F). \tag{B-52}$$

Then, the linearized form of the fourth term is obtained as

$$\delta(\Gamma_{5M}^K \Gamma_{5K}^M) = 8A'F'. \tag{B-53}$$

As a final step we will add these four terms so that we get the linearized form of R_{55} as

$$\delta R_{55} = -4F'' - 2e^{2A}\square F + 16A'F'. \tag{B-54}$$

At this stage, we will derive the linearized form of the source term. We start with the part of the action (see eq. 4.78) including the source term and taking $\sqrt{g_{55}}$ as $\sqrt{g_{55}} = 1 + 2F$. The metric variation on this action gives

$$\begin{aligned}
T^{MN} &= \frac{1}{2}g^{MN}\left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] - \frac{1}{2}\partial^M\phi\partial^N\phi \\
&- \frac{1}{2(1+2F)}g_\mu^M g_\nu^N g^{\mu\nu} \sum_i \lambda_i(\phi)\delta(y-y_i). \tag{B-55}
\end{aligned}$$

Since $T = g_{MN}T^{MN}$ we get

$$T = \frac{5}{2}[\frac{1}{2}(\partial\phi)^2 - V(\phi)] - \frac{1}{2}(\partial\phi)^2 - \frac{2}{1+2F} \sum_i \lambda_i(\phi)\delta(y-y_i). \quad (\text{B-56})$$

Using the form of Einstein equation given in eq. B-1, and the eqs. B-55 and B-56 to get

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}g_{\mu\nu}T, \quad (\text{B-57})$$

we obtain

$$\begin{aligned} \tilde{T}_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}[\frac{1}{2}(\partial\phi)^2 - V(\phi)] - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2(1+2F)}g_{\mu\nu} \sum_i \lambda_i(\phi)\delta(y-y_i) \\ &\quad - \frac{5}{6}g_{\mu\nu}[\frac{1}{2}(\partial\phi)^2 - V(\phi)] + \frac{1}{6}g_{\mu\nu}(\partial\phi)^2 + \frac{2}{3(1+2F)}g_{\mu\nu} \sum_i \lambda_i(\phi)\delta(y-y_i) \\ &= \frac{1}{3}g_{\mu\nu}V(\phi) - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{6(1+2F)}g_{\mu\nu} \sum_i \lambda_i(\phi)\delta(y-y_i), \end{aligned} \quad (\text{B-58})$$

where

$$\begin{aligned} \phi(x, y) &= \phi_0(y) + \varphi(x, y), \\ V(\phi) &= V(\phi_0) + \varphi V'(\phi_0), \\ \lambda_i(\phi) &= \lambda_i(\phi_0) + \varphi \lambda'_i(\phi_0), \end{aligned} \quad (\text{B-59})$$

for small fluctuations. Substituting the equations in eq. B-59 into $\tilde{T}_{\mu\nu}$ we have

$$\begin{aligned} \tilde{T}_{\mu\nu} &\cong \frac{1}{3}e^{-2A}(1-2F)\eta_{\mu\nu}[V(\phi_0) + \varphi V'(\phi_0)] - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi \\ &\quad + \frac{1}{6}e^{-2A}(1-2F)^2\eta_{\mu\nu} \sum_i [\lambda_i(\phi_0) + \varphi \lambda'_i(\phi_0)]\delta(y-y_i). \end{aligned} \quad (\text{B-60})$$

The linearized form of the source term $\tilde{T}_{\mu\nu}$ is obtained as

$$\begin{aligned}\delta\tilde{T}_{\mu\nu} = & \frac{1}{3}e^{-2A}\eta_{\mu\nu}[\varphi V'(\phi_0) - 2V(\phi_0)F] \\ & + \frac{1}{6}e^{-2A}\eta_{\mu\nu}\sum_i\left[\frac{\partial\lambda_i(\phi_0)}{\partial\phi}\varphi - 4\lambda_i(\phi_0)F\right]\delta(y-y_i).\end{aligned}\tag{B-61}$$

Now, let us derive the equation for the source term $\tilde{T}_{\mu 5}$ by using

$$\tilde{T}_{\mu 5} = T_{\mu 5} - \frac{1}{3}g_{\mu 5}T.\tag{B-62}$$

Since $g_{\mu 5} = 0$ we get

$$\tilde{T}_{\mu 5} = T_{\mu 5}.\tag{B-63}$$

Using the eq. B-55 we obtain $\tilde{T}_{\mu 5}$ as

$$\begin{aligned}\tilde{T}_{\mu 5} &= -\frac{1}{2}\partial_\mu\phi\partial_5\phi \\ &= -\frac{1}{2}[\partial_\mu(\phi_0(y) + \varphi(x, y))\partial_5(\phi_0(y) + \varphi(x, y))].\end{aligned}\tag{B-64}$$

Then, the linearized form of $\tilde{T}_{\mu 5}$ reads

$$\delta\tilde{T}_{\mu 5} = -\frac{1}{2}\phi'_0\partial_\mu\varphi.\tag{B-65}$$

Finally, T_{55} can be obtained by using the eq. B-55

$$T_{55} = -\frac{1}{2}(1 + 2F)^2\left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] - \frac{1}{2}\partial_5\phi\partial_5\phi.\tag{B-66}$$

Then, the source term \tilde{T}_{55} becomes

$$\begin{aligned}
\tilde{T}_{55} &= T_{55} - \frac{1}{3}g_{55}T \\
&= -\frac{1}{2}(1+2F)^2\left[\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] - \frac{1}{2}\partial_5\phi\partial_5\phi \\
&\quad + \frac{1}{2}(1+2F)^2\left[\frac{1}{2}(\partial\phi)^2 - \frac{5}{3}V(\phi)\right] \\
&\quad - \frac{2}{3}(1+2F)\sum_i\lambda_i(\phi)\delta(y-y_i).
\end{aligned} \tag{B-67}$$

Making the simplifications and substituting the eqs. B-59 we get

$$\begin{aligned}
\tilde{T}_{55} &\cong -\frac{1}{3}(1+4F+4F^2)[V(\phi_0) + \varphi V'(\phi_0)] \\
&\quad - \frac{1}{2}\partial_5[\phi_0(y) + \varphi(x, y)]\partial_5[\phi_0(y) + \varphi(x, y)] \\
&\quad - \frac{2}{3}(1+2F)\sum_i[\lambda_i(\phi_0) + \varphi\lambda'_i(\phi_0)]\delta(y-y_i).
\end{aligned} \tag{B-68}$$

The linearized form is obtained as

$$\begin{aligned}
\delta\tilde{T}_{55} &= -\frac{4}{3}V(\phi_0)F - \frac{1}{3}\varphi V'(\phi_0) - \varphi'\phi'_0 \\
&\quad - \frac{2}{3}\sum_i\left[\frac{\partial\lambda_i(\phi_0)}{\partial\phi}\varphi + 2\lambda_i(\phi_0)F\right]\delta(y-y_i).
\end{aligned} \tag{B-69}$$

APPENDIX C

In this Appendix we present the formulation for the spin connection which we use in our calculations (see [70]). A natural basis for the tangent space T_p at a point p is given by the partial derivatives with respect to the coordinates at that point, $\hat{e}_{(\mu)} = \partial_\mu$. Similarly, a basis for the cotangent space T_p^* is given by of the coordinate functions, $\hat{\theta}_{(\mu)} = dx^\mu$. Let us imagine that at each point in the manifold there exists a set of orthonormal basis vectors $\hat{e}_{(a)}$ ¹. If the canonical form of the metric is written η_{ab} , the inner product of our basis vectors should be

$$\hat{e}_{(a)} \cdot \hat{e}_{(b)} = \eta_{ab}. \quad (\text{C-1})$$

Thus, in Lorentzian space η_{ab} represents the Minkowski metric, while in a space with positive definite metric it represents Euclidean metric. We can express our old basis vectors of tangent space in terms of the new ones as

$$\hat{e}_{(\mu)} = e_\mu^a \hat{e}_{(a)}, \quad (\text{C-2})$$

where the components e_μ^a are called as vielbeins form an invertible $n \times n$ matrix. Their inverse is denoted by switching the indices to obtain e_a^μ , which satisfy

$$e_a^\mu e_\nu^a = \delta_\nu^\mu \quad ; \quad e_\mu^a e_b^\mu = \delta_b^a. \quad (\text{C-3})$$

Multiplying the eq. C-2 with e_a^μ from left

$$\hat{e}_{(a)} = e_a^\mu \hat{e}_{(\mu)}, \quad (\text{C-4})$$

¹indexed in Latin letter rather than Greek, to remind us that they are not related to any coordinate system.

Then, in terms of the inverse vielbeins the eq. C-1 becomes

$$\begin{aligned}
\eta_{ab} &= \hat{e}_{(a)} \cdot \hat{e}_{(b)} \\
&= e_a^\mu \hat{e}_{(\mu)} \cdot e_b^\nu \hat{e}_{(\nu)} \\
&= g_{\mu\nu} e_a^\mu e_b^\nu.
\end{aligned} \tag{C-5}$$

We see that the components of the metric tensor in the orthonormal basis are just those of flat metric, η_{ab} . Multiplying this with $e_\mu^a e_\nu^b$ we get

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}. \tag{C-6}$$

Thus, vielbeins are squareroot of the metric. Similarly in T_p^* , $\theta^{(a)}$ are the orthonormal basis of one-forms. Choosing

$$\theta^{(a)} \hat{e}_{(b)} = \delta_b^a = e_\mu^a e_b^\mu, \tag{C-7}$$

it is an immediate consequence that the orthonormal one-forms are related to the coordinate-based $\hat{\theta}_{(\mu)}$ by

$$\hat{\theta}_{(\mu)} = e_a^\mu \hat{\theta}_{(a)}. \tag{C-8}$$

Similarly,

$$\hat{\theta}_{(a)} = e_\mu^a \hat{\theta}_{(\mu)}. \tag{C-9}$$

Any vector V written in the coordinate basis as $V = V^\mu \hat{e}_\mu$ can be expressed as in terms of its orthonormal basis as $V = V^a \hat{e}_a$. Since $V^\mu \hat{e}_\mu = V^a \hat{e}_a$ we obtain a relation between the sets of components as

$$V^a = e_\mu^a V^\mu. \tag{C-10}$$

So the vielbeins allow us to pass from Latin to Greek indices and back. In the same manner multi index tensor V_b^a can be written as

$$V_b^a = e_\mu^a V^\mu_b = e_b^\nu V^a_\nu = e_\mu^a e_b^\nu V^\mu_\nu. \tag{C-11}$$

Nice property of tensors is that, we can go on to refer multi index tensors in terms of mixed components.

Now, we have a set of basis vectors \hat{e}_a and θe_a which are non-coordinate bases. Thus, they can be changed independently of the coordinates provided the orthonormality property defined in eq. C-1 is preserved. In Euclidean signature metric the transformations that preserve orthonormality condition are orthogonal transformations whereas in Lorentz signature metric they are Lorentz transformations. We therefore consider changes of basis of the form

$$\hat{e}_a \rightarrow \hat{e}_{a'} = \Lambda_{a'}^a(x) \hat{e}_a, \quad (\text{C-12})$$

where $\Lambda_{a'}^a(x)$ represent position dependent transformations at each point in space which leave the canonical form of metric unaltered such that

$$\Lambda_{a'}^a(x) \Lambda_{b'}^b(x) \eta_{ab} = \eta_{a'b'}. \quad (\text{C-13})$$

In flat space, we call these matrices inverse Lorentz transformations. We also have ordinary Lorentz transformations, $\Lambda_a^{a'}(x)$ to transform one-forms. So, we now have freedom to perform a Lorentz transformation at every point in space. These are called local Lorentz transformations (LLT).

The covariant derivative of a tensor is given by its partial derivative plus the connection terms, one for each index. The connection terms in our ordinary formalism involve the tensor and connection coefficients $\Gamma_{\mu\nu}^\lambda$ whereas in non-coordinate basis connection coefficients are replaced by spin connection, denoted by $w_\mu^a{}_b$. Each Latin index gets a factor of the spin connection as

$$\nabla_\mu X^a{}_b = \partial_\mu X^a{}_b + w_\mu^a{}_c X^c{}_b - w_\mu^c{}_b X^a{}_c. \quad (\text{C-14})$$

Now let us try to find a relation between the spin connection, the vielbeins and connection coefficients using the property that a tensor should be independent of the way it is written. For simplicity, we will consider the covariant derivative of

a vector X . Its covariant derivative in a purely coordinate basis is given by

$$\begin{aligned}\nabla X &= (\nabla_\mu X^\nu) dx^\mu \otimes \partial_\nu \\ &= (\partial_\mu X^\nu + \Gamma_{\mu\lambda}^\nu X^\lambda) dx^\mu \otimes \partial_\nu,\end{aligned}\tag{C-15}$$

and in a mixed basis it is written as

$$\begin{aligned}\nabla X &= (\nabla_\mu X^\nu) dx^\mu \otimes \hat{e}_{(a)} \\ &= (\partial_\mu X^a + w_\mu^a{}_b X^b) dx^\mu \otimes \hat{e}_{(a)} \\ &= (\partial_\mu (e_\nu^a X^\nu) + w_\mu^a{}_b e_\lambda^b X^\lambda) dx^\mu \otimes \hat{e}_{(a)} \\ &= (e_\nu^a \partial_\mu X^\nu + X^\nu \partial_\mu e_\nu^a + w_\mu^a{}_b e_\lambda^b X^\lambda) dx^\mu \otimes e_a^\sigma \partial_\sigma \\ &= e_a^\sigma (\partial_\mu X^\nu + e_\nu^a \partial_\mu X^\nu + X^\nu \partial_\mu e_\nu^a + w_\mu^a{}_b e_\lambda^b X^\lambda) dx^\mu \otimes \partial_\sigma \\ &= (\delta_\nu^\sigma \partial_\mu X^\nu + e_a^\sigma X^\nu \partial_\mu e_\nu^a + e_a^\sigma w_\mu^a{}_b e_\lambda^b X^\lambda) dx^\mu \otimes \partial_\sigma \\ &= (\partial_\mu X^\sigma + e_a^\sigma X^\nu \partial_\mu e_\nu^a + e_a^\sigma w_\mu^a{}_b e_\lambda^b X^\lambda) dx^\mu \otimes \partial_\sigma.\end{aligned}\tag{C-16}$$

Let $\sigma \rightarrow \nu$ and $\nu \rightarrow \lambda$

$$\nabla X = (\partial_\mu X^\nu + e_a^\nu (\partial_\mu e_\lambda^a) X^\lambda + e_a^\nu e_\lambda^b w_\mu^a{}_b X^\lambda) dx^\mu \otimes \partial_\nu.\tag{C-17}$$

Comparing the eqs. C-15 and C-17 one can conclude that

$$\Gamma_{\mu\lambda}^\nu = e_a^\nu (\partial_\mu e_\lambda^a) X^\lambda + e_a^\nu e_\lambda^b w_\mu^a{}_b.\tag{C-18}$$

Multiplying this with $e_b^\lambda e_\nu^a$ we get

$$w_\mu^a{}_b = e_\nu^a e_b^\lambda \Gamma_{\mu\lambda}^\nu - e_b^\lambda \partial_\mu e_\lambda^a.\tag{C-19}$$

Now let us look the covariant derivative of a vielbein

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + w_\mu^a{}_b e_\nu^b - \Gamma_{\mu\nu}^\alpha e_\alpha^a.\tag{C-20}$$

Substituting the equation for $\Gamma_{\mu\lambda}^\nu$ into the covariant derivative of the vielbein we

get

$$\nabla_\mu e_\nu^a = 0. \tag{C-21}$$